Due Apr. 23, Tuesday.

Problem 1 (15 points) Using the Sterling's formula (Eq. 4.34) and the Taylor expansion of the logarithm, prove only one of the following two statements (a) and (b) (why only one of the two? – just to save some time).

(a) The binomial distribution (Eq. 3.25) becomes a Gaussian distribution, in the limit of $Nh \to \infty$, $N(1-h) \to \infty$. Hint: Write $n = Nh + x$, and note that $|x| \ll Nh$ when Nh is large. Take the logarithm of the binomial distribution, obtain the leading order expansion for x.

(b) The Poisson distribution Eq. 3.49 becomes a Gaussian distribution, in the limit of $\lambda = Nh \to \infty$. Hint: Write $y = x - \lambda$, and note that $|y| \ll \lambda$ when λ is large. Take the logarithm of the Poisson distribution, obtain the leading order expansion for x.

Problem 2 (20 points) Consider a Lorentzian distribution function Eq. 3.55:

$$p(\omega) = \frac{1}{\pi} \frac{\Gamma}{(\omega - \omega_R)^2 + \Gamma^2} \qquad \Gamma > 0.$$

(a) By contour integration, or any other method that you see fit, prove that $\int_{-\infty}^{\infty} d\omega\, p(\omega) = 1$. Note that the mean of this distribution is not defined, strictly speaking. However, the median is well-defined, and it is ω_R. The full width at half maxima (FWHM) is also well-defined, 2Γ, while the standard deviation is not.

(b) Show that the characteristic function $\tilde{p}(t) \equiv \langle e^{-i\omega t} \rangle$ does exist, despite the fact that most moments and cumulants are ill-defined. Find it (by contour integration or some other method that you see fit).

(c) Suppose that the above distribution is an approximate description of the frequency distribution of photons from a certain atom. Furthermore, let us assume that we have N atoms, each of which can be different from one another. Then, we can consider the following sum variable

$$\omega = \omega_1 + \omega_2 + \ldots + \omega_N$$

where each ω_j ($j = 1, \ldots, N$) satisfies the following distribution separately (and thus, independently of one another)

$$p(\omega_j) = \frac{1}{\pi} \frac{\Gamma_j}{(\omega_j - \omega_{R,j})^2 + \Gamma_j^2} \qquad \Gamma_j > 0.$$

Find the probability distribution for ω, and show that it is a Lorentzian distribution for any N value, demonstrating that the central limit theorem does not apply here. What is the median value of ω? How about the FWHM?

[Note for optional reading:] Here, we considered ω to be an unrestricted real number. One might point out that, in an actual experiment, frequencies, such as ω_R and ω, must be non-negative, by definition. How can be invoke negative frequencies? (1) As a general rule, $\Gamma \ll \omega_R$, and so extending the distribution to negative ω has only small consequences, if any. (2) A many-body Green's function *does* have a negative frequency part, which corresponds to the time-inverted process of the positive frequency part, and the spectral weight, which is proportional to the imaginary part of the Green's function and is approximately given by a Lorentzian function in many cases, normalizes to one, only if integrated from $-\infty$ to ∞.

Problem 3 (10 points) Assume that typographical errors committed by a typesetter are completely random. Suppose that a book of 1000 pages contain 1000 such errors.

(a) What is the probability that a page will contain no error?

(b) What is the probability that a page will contain three errors

Problem 4 (10 points) A mirror is plated with gold by evaporation. The evaporation method works by heating a gold wire in a high vacuum. Gold atoms fly off in all directions and a portion of them sticks to the glass, or to other atoms already on the glass plate. Assume that each column of deposited atoms is independent of neighboring columns, and that the average deposition rate is d layers per second.

(a) What is the probability of m atoms deposited at a site after a time t? What fraction of the glass is not covered by any gold atoms?

(b) What is the variance in the thickness?

Problem 5 (15 points) By using the cluster expansion, find the following quantities in terms of cumulants and, when applicable, joint cumulants.

(a) $\langle x^5 \rangle$

(b) $\langle x^6 \rangle$

(c) $\langle x_1^3 x_2^2 \rangle$

Problem 6 (15 points) Consider a container containing an ideal gas. The container is kept at a constant temperature by the environment. Initially, the gas occupies half the container, due to a partitioning wall at the center of the container. However, the partitioning wall has an externally controllable mechanism by which it can open a large opening at its center suddenly (a bit like a camera aperture). As this mechanism is activated, the gas quickly expands to the entire volume of the container, doubling the volume. Now, one can ask – what is the probability that the state of the gas will evolve back to its initial state, in which it occupied only one partition? Since the microscopic laws that govern the dynamics of the gas (electromagnetism and gravity) are time-reversal invariant, such an evolution in time *is clearly a possibility.* The question is *how* possible is it?

(a) A simple way to estimate this probability is to calculate at a given time, the probability of a single molecule to be in half the volume, and then exponentiate the number N times, assuming all molecules are independent, and N is the number of molecules. Estimate the probability this way, using $N = 200$, a smallish number that is still much larger than 1, and $N = 10^{23}$, for a typical macroscopic amount.

(b) Note that a small probability does not necessarily mean that the event will not happen. A more realistic question is in what time scale would the event occur once? To estimate *the lower bound of* this time scale, we must know the "relaxation time scale" – the time scale in which the system readjusts locally. The relaxation time for a gas system in a typical ambient condition is well-known: it is about 0.1 ns. Estimate the lower bound of the time scale, in unit of years, in which the "spontaneous volume contraction" will occur for $N = 200$ and $N = 10^{23}$.

(c) According to recent studies (Phys. Rev. E, vol. 60, 2721 (1999) and subsequent papers citing it), the following identity holds.

$$\frac{P_F(\Delta S)}{P_R(-\Delta S)} = e^{\frac{\Delta S}{k_B}}$$

where $P_F(\Delta S)$ is the probability for a forward process involving ΔS, and $P_R(-\Delta S)$ is the probability for its time-reversed process. If we take the forward process as the expansion process after the partitioning wall opened, then clearly P_F must be close to, or practically equal to, 1. Assuming ideal gas, for which the entropy function is well-known (Eq. T1.5.1a), calculate $P_R(-\Delta S)$, and compare your answer with the answer for part (a).

Problem 7 (30 points) *Poincaré period.* This problem will show you how to estimate the Poincaré period.

(a) Consider two oscillatory functions $\psi_1 = \exp(-i\omega_1 t)$ and $\psi_2 = \exp(-i\omega_2 t)$. We assume that $\omega_1 \neq \omega_2$ and furthermore that they are incommensurate with each other. We can interpret ψ_1 and ψ_2 as representing two points on a unit radius circle. As written, at $t = 0$, the two points coincide. After some time the two points will be at a finite angular distance away from each other, since $\omega_1 \neq \omega_2$. The question one can ask is: how long would it take for the two points to come back to the zero angle position within a small tolerance $\Delta\theta$? [Note that $\Delta\theta$ here can identified with $\epsilon/\sqrt{2}$ in Eq. 5.22.] Now, the first point has the period of $T_1 = 2\pi/\omega_1$ and it spends time $t_1 = T_1\Delta\theta/(2\pi)$ in the "sweet zone" ($\Delta\theta$) per revolution. For the second point, $T_2 = 2\pi/\omega_2$ and $t_2 = T_2\Delta\theta/(2\pi)$. We define *coincidence* as the state in which both of these points are in the sweet zone. How frequently this coincidence occurs is characterized by the recurrence period. In order to figure out the recurrence period, it is helpful to consider the following function $t_{12}(\theta_1)$. Without loss of generality, assume $T_1 > T_2$. Consider the moment when point 2 is just entering the sweet zone. Point 1 can be anywhere on the circle at this moment. Let its position be θ_1 ($-\pi < \theta_1 \leq \pi$). As a function of θ_1, we can calculate how long, if at all, the two points will be in the sweet zone together before point 2 leaves the sweet zone after time t_2 has passed. This *duration of coincidence* as a function of θ_1 is defined as $t_{12}(\theta_1)$. Obtain this function.

(b) (This part is optional, since it is not absolutely necessary. However, you must read this part, and understand the final result. See the next part for a quicker way of proving the final result.) With the function $t_{12}(\theta_1)$ thus obtained, one can calculate the *average coincidence lifetime* τ_C as

$$\tau_C = \frac{\text{the average of } t_{12}}{\text{the probability of the coincidence}}$$

Note that the probability of the coincidence is proportional to the extent of θ_1 in which $t_{12}(\theta_1)$ is non-zero. τ_C represents the average duration of the coincidence given the condition that a coincidence happened. Show that

$$\frac{1}{\tau_C} = \frac{1}{t_1} + \frac{1}{t_2}$$

(c) Explain the last result by considering the following. Suppose that a coincidence is observed at time t. Consider the probability that this coincidence will be broken within the next infinitesimal time interval dt. By definition, this probability is dt/τ_C. It can also be expressed in terms of t_1, t_2, and dt. Show clearly that this consideration leads exactly to the above result ("Matthiessen's rule").

(d) Now generalize to the case when there are M oscillators with mutually incommensurate periods. By using the same argument, prove that

$$\frac{1}{\tau_C} = \sum_{i=1}^{M} \frac{1}{t_i}$$

where $t_i \equiv T_i \Delta\theta/(2\pi)$ and T_i is the period of the i-th oscillator.

(e) In the limit of running the system to the infinity of time, we can express the probability of coincidence in two ways. The first way is simple: $\left(\frac{\Delta\theta}{2\pi}\right)^M$. The second way is τ_C/T_{pc}, where T_{pc} is the recurrence period, or the Poincaré cycle. By equating these two quantities, and using the results above, show that

$$T_{pc} = \frac{2\pi}{\omega_{ave}} \frac{1}{M} \exp\left[(M-1)\log\frac{2\pi}{\Delta\theta}\right]$$

where ω_{ave} is the average frequency of the M oscillators.

(f) As an example of a classical mechanical system, consider normal modes of linearly coupled oscillators as applicable to the molecular vibrations in a one dimensional solid. The typical frequency is the Debye frequency so that $2\pi/\omega_{ave} \sim 10^{-12}$ sec. If there are 101 atoms in a linear chain, then $M = 100$ (excluding one translational mode that has zero frequency). Evaluate and discuss the value of T_{pc}.

(g) As a quantum mechanical example, we may consider an electron gas. When this system goes out of equilibrium (due to an external field or a temperature gradient), it relaxes to a local equilibrium by collision processes. This time scale is what we discussed in class as τ_X. By a conservative estimate, $\tau_X \sim 1$ ns, while it is typically *much* shorter than 1 ns. Note that τ_X gives a lower bound for the energy width ΔE through the Heisenberg uncertainty relation. To estimate the corresponding lower bound of the number of frequency modes, one must divide ΔE by the energy increment of the system. This is roughly[1] $\hbar^2 4\pi^2/(2mL^2)$ where L is the sample size or the mean free path $v_F\tau_X$, whichever is larger. So, for $\tau_X \sim 1$ ns, we have $L \sim v_F\tau_X \sim 1$ mm at least, if $\hbar v_F \sim 5$ eV Å. Estimate M based on these parameters and the corresponding T_{pc}. Discuss the relationship between the irreversibility time scale (note: τ_X is the time scale in which the local equilibrium is established) and T_{pc}, if any. [Hint: $\hbar c = 1973$ eV Å, and $mc^2 = 0.511$ MeV would be useful.]

Problem 8 (15 points) The *entropy function* of a PDF is defined as

$$s = -\sum_j p_j \log p_j \qquad\qquad j = 1, 2, ..., N$$

where the random variable is assumed to be discrete, and the extension to a continuous variable is obvious (while such an extension is not without an issue – see LN 5). Note that $p_j \geq 0$, by definition. If $p_j = 0$, then it contributes to 0 to the sum, since $0^+ \log 0^+ = 0$.

(a) Show that $k_B{\cdot}s$ becomes the "Boltzmann entropy" $S = k_B \log N$, if p_j is uniformly distributed over N possible outcomes.

(b) Prove that for any non-uniform distribution, s is smaller than the one in the previous part. [Hint: Here, the Lagrange multiplier method is the way to go. As there is a constraint, $\sum_j p_j = 1$, on the N variables $\{p_j\}$, one examines the extrema of the function, $f(\{p_j\}) = -\sum_j p_j \log p_j - \alpha(1 - \sum_j p_j)$. Get $N+1$ coupled equations to solve, N by differentiating f and 1 from the constraint equation itself, and solve them for $N+1$ unknowns ($\{p_j\}$ and α).]

[1] This estimate is based on the independent particle picture. In an interacting system, this estimate will tend to become smaller, e.g., due to the formation of "heavy fermions."