# Notes for Lecture 16

# Landau theory

We consider the Ising model from the point of view of the variational principle, as well as other points of view, and introduce the Landau theory of phase transition.

## 16.1 Ising model (cont.)

We continue the study of the ferromagnetic Ising model, and develop important ways of thinking about it.

Let us start by checking some simple definitions.

$$M \equiv \sum_i \overline{\langle \sigma_i \rangle} = N \overline{\langle \sigma_i \rangle} \qquad \text{magnetization in unit of } \mu_B \tag{16.1}$$

$$m \equiv x \equiv \frac{M}{N} \qquad N = \text{the total number of spins} \tag{16.2}$$

Here, we just introduced the symbol $m$ for the magnetization per spin, the same as $x$ that we used in the last lecture. From this lecture note, we shall use $m$ exclusively. We continue to use $h$ as the field that drives $m$. Normally, we would use $H$ for the magnetic field, but we use $h$ for avoiding conflict with the Hamiltonian symbol.

### 16.1.1 Gibbs free energy

Continuing the mean field theory, let us find what the Gibbs free energy is.

Let us start with a question: for the given the partition function of Eq. 15.24, what would the quantity $-k_BT \log Z$ represent? The answer is the Gibbs free energy, *not* the Helmholtz free energy![1] So, the partition function that we've been considering is the *Gibbs* partition function (cf. lecture 7). Why is this? Recall from the context of the $PV$ system, the Gibbs free energy refers to the case when $T$ and $P$ are fixed by the reservoir, or, in other words, are taken as independent variables to describe the system. In the magnetic system, the "force" variable is $h$ (we will continue to use a lower case to mean the magnetic field, to avoid confusion with the Hamiltonian) and the "displacement" variable is $M$. If one looks back on the calculation that we did in the last lecture, one can see that we did not consider $M$ as fixed. Rather, we considered $h$ as fixed (0 for example, if the system is under zero field). This means that we have been considering the Gibbs canonical ensemble for the Ising model.

With this understanding firmly in place, from Eq. 15.24, we get

$$G_i(T,h) = -k_BT \log\cosh(\beta h_i \mu_B) - k_BT \log 2 \quad \text{not quite right – see below} \tag{16.3}$$

$$G(T,h) = -Nk_BT \log\cosh(\beta h_i \mu_B) - Nk_BT \log 2 \quad \text{not quite right – see below} \tag{16.4}$$

Using the relation (from Eq. 15.23 and the discussion below it)

$$h_i \mu_B = Jzm + \mu_B h \tag{16.5}$$

we get

$$G(T,h) = -Nk_BT \log\cosh\left(\beta Jzm + \beta\mu_B h\right) - Nk_BT \log 2 \quad \text{not quite right – see below} \tag{16.6}$$

In this expression, we must consider $m$ as $m(T,h)$. All this seems fine, until we realize a problem! If one takes $-\left(\frac{\partial G}{\partial h}\right)_T$, one must get $M\mu_B = mN\mu_B$, which must agree with Eq. 15.26. But, by doing this, we get

$$m\mu_B = \tanh(\beta Jzm + \beta\mu_B h)\left(Jz\frac{\partial m}{\partial h} + \mu_B\right) \quad \text{definitely incorrect!} \tag{16.7}$$

instead, which is *not* equal to Eq 15.26. What is wrong? The answer is the neglected term in Eq. 15.21. In the last lecture, we were focusing on the single spin dynamics alone, and so we could ignore the last term of that equation, since it did not involve any dynamic variable (it was "just a constant"). That term could be neglected, since it did not have any "value" for that lecture. However, from the point of view of the *total* Gibbs free energy, the last term, which is $\frac{JNz}{2}m^2$, definitely cannot be ignored and must be included ($Nz/2$ is the total number of the nearest neighbor pairs). So, the Gibbs free energy must be written as

$$G(T,h) = -Nk_BT \log\cosh(\beta(Jzm + \mu_B h)) + \frac{JNz}{2}m^2 - Nk_BT \log 2 \tag{16.8}$$

[1] To be consistent with the notation of LN 7, then we should have used $\mathcal{Z}$. Nevertheless, this notation difference is within the fluctuation in the literature, and so we will continue to use $Z$.

Now, applying $mN\mu_B = -\left(\frac{\partial G}{\partial h}\right)_T$, one gets

$$\left(1 + \frac{\partial m}{\partial h}\frac{Jz}{\mu_B}\right)\left(m - \tanh(\beta Jzm + \beta\mu_B h)\right) \;=\; 0 \tag{16.9}$$

The quantity $\left(1 + \frac{\partial m}{\partial h}\frac{Jz}{\mu_B}\right)$ is always positive for a ferromagnetic system (since the susceptibility $\frac{\partial m}{\partial h}$ is positive[2]) and thus we conclude

$$m \;=\; \tanh(\beta Jzm + \beta\mu_B h) \tag{16.10}$$

the same correct result as we got in the last lecture.

### 16.1.2 Variational principle

As we discussed in the last lecture, the above equation has three solutions if $T < T_c$ where

$$T_c \;=\; \frac{Jz}{k_B} \tag{16.11}$$

This temperature is (properly) the "critical temperature" as we shall see later on in this lecture. It is also called the Curie temperature in the case of the ferromagnetic model currently under consideration.

Now it is time to consider which of these three solutions are physical. The way to do it is to consider the Gibbs free energy of Eq. 16.8 as the "would be" Gibbs free energy where $m$ is the variational parameter that minimizes the "would be" Gibbs free energy. This is in perfect analogy with what we did in Section 9.3 (and Section 9.4). There we considered $N$ as the variational parameter for the would be grand potential: likewise, here we consider $m$ as the variation parameter for the would be Gibbs free energy. With this understanding firmly in place, we will simply forgo with the "would be" designation in the following discussion, for brevity. We will, in fact, forgo with the "Gibbs" designation also, since there is no room for confusion.

Let us express the free energy in terms of dimensionless variables

$$t \;\equiv\; \frac{T - T_c}{T_c} \;=\; \frac{k_B T}{Jz} - 1 \tag{16.12}$$

$$h_r \;\equiv\; \frac{\mu_B h}{Jz} \tag{16.13}$$

[2]It is, in particular, $+\infty$ below the transition.

Then, we get, for the free energy per spin,

$$g(T,h) \equiv \frac{G(T,h)}{N} = -k_B T \log\cosh\left(\frac{m+h_r}{1+t}\right) + \frac{k_B T}{2(1+t)} m^2 - k_B T \log 2 \quad (16.14)$$

and thus we get, equivalently to Eq. 16.8,

$$g(T,h) = k_B T \left[\frac{m^2}{2(1+t)} - \log\cosh\left(\frac{m+h_r}{1+t}\right) - \log 2\right] \quad (16.15)$$

We are particularly interested in the behavior of the free energy near $T_c$, i.e. when $m$, $|t|$ and $|h_r|$ are all small. In particular, we will be interested in the response of the system in the leading order of $|t|$ and $|h_r|$: this means expanding the free energy up to linear order in these parameters. However, note that $m$ is not an input parameter at all. We do not know *a priori* how small it is, and so we must expand the free energy in enough orders of $m$ so that our solution is physically sensible. For sensible results, we will find that keeping terms up to order $m^4$ is sufficient. Then, using $\cosh(\delta) \approx \frac{1}{2}\delta^2 + \frac{1}{24}\delta^4$, and $\log(1 + \frac{1}{2}\delta^2 + \frac{1}{24}\delta^4) \approx \frac{1}{2}\delta^2 - \frac{1}{12}\delta^4$, we get $g(T,h)\beta \approx \frac{m^2}{2(1+t)} - \frac{1}{2}\left(\frac{m+h_r}{1+t}\right)^2 + \frac{1}{12}\left(\frac{m+h_r}{1+t}\right)^4$, which leads to the result that

$$g(T,h) \approx k_B T \left[\frac{t}{2} m^2 + \frac{1}{12} m^4 - m h_r - \log 2\right] \quad (16.16)$$

where for $|t|$ and $|h_r|$ we have kept only the linear order terms that involve the lowest order terms of $m$.

This is an important result[3] that shows that, if $T < T_c$ ($t < 0$) and $h = 0$, then $m = 0$ is a maximum of the free energy – thus, unphysical – since the coefficient of the second order term is negative. Instead, two physical solutions that correspond to the minima of $G$ emerge, since as $m$ increases the next order term ($O(m^4)$) forces the free energy to bounce back. This is the proof of the statement that we made in the previous lecture note that, in Eq. 15.28, $m = \pm\bar{\sigma}$ are physical solutions but $m = 0$ is not. For $m \approx 0$ and $h = 0$, the physical values of $m$ are obtained perturbatively, from the two terms of the above equation, by setting $\frac{\partial g}{\partial m} = 0 \propto tm + m^3/3 = 0$ as

$$m = \pm\sqrt{-3t} \approx \pm\sqrt{3}\left(1 - \frac{T}{T_c}\right)^{1/2} \qquad \text{for } T < T_c \text{ and } T \approx T_c \quad (16.17)$$

[3] In this note, this result has been derived from the mean field theory that we set up in the last lecture. An equivalent way to get this result is to use the so-called "Bragg-Williams approximation," which is very much similar in spirit to the mean field approximation that we have employed here but is different enough in the method to give a somewhat different expression for $g$, whose expansion near $m, |h_r|, |t|$ is, nevertheless, identical with what we got here. This other method is discussed in standard textbooks, e.g. Plischke and Bergersen, or Huang.

are possible physical values for an equilibrium state. On the other hand,

$$m \;=\; 0 \qquad\qquad \text{for } T > Tc \tag{16.18}$$

is the only physical state for equilibrium above the transition temperature. Of course, these results could also have been obtained straight from the solution for $m$ (Eq. 16.10), which can be re-written as

$$m \;=\; \tanh\left(\frac{m+h_r}{1+t}\right) \tag{16.19}$$

by setting $h = 0$ and using the expansion $\tanh\delta \approx \delta - \delta^3/3$.

In any case, Eq. 16.17 means that $M \propto (T - T_c)^{1/2}$ near but below $T_c$. In the critical phenomena lingo, this means that[4] the so-called "$\beta$ exponent" is 1/2. This mean field result is incorrect in general. The correct value is 1/8 in two dimensions, and $\approx 0.313$ for three dimensions.

### 16.1.3 Curie-Weiss susceptibility

Let us consider the magnetic susceptibility per spin, in unit of $\mu_B$ per field strength (gauss, e.g.)

$$\chi(T) \;\equiv\; \frac{\partial m}{\partial h} \tag{16.20}$$

The susceptibility can be obtained from Eq. 16.19 by taking the derivative on $h$.

$$\chi \;=\; \frac{1}{\cosh^2\left(\frac{m+h_r}{1+t}\right)}\left[\frac{\chi}{(1+t)} + \frac{\mu_B}{Jz(1+t)}\right] \tag{16.21}$$

Assuming low field, and assuming that $m = 0$ (above $T_c$), we can approximate $\cosh\left(\frac{m+h_r}{t+1}\right) \approx 1$, and we get

$$\chi \;=\; \frac{\mu_B}{Jz}\cdot\frac{1}{t} \;=\; \frac{\mu_B}{k_B}\cdot\frac{1}{T-T_c} \qquad\qquad T > T_c \tag{16.22}$$

This is the "Curie-Weiss law" ($\chi \propto 1/(T-T_c)$) for a ferromagnet above the transition temperature. A special case, $J = 0$, corresponds to the $T_c = 0$ case, which gives the Curie's law $\chi \propto 1/T$. For Curie's law to apply, one does not need a lattice

[4] The exponent $\beta$ that defines $M \propto (T - T_c)^\beta$ has nothing to do with the temperature. It is unfortunate that we use $\beta$ for two things, but, if necessary, please use the context to figure out which $\beta$ we are talking about.

of spins. Any number of non-interacting spins will give rise to Curie's law. For instance, a small amount of magnetic impurities give a large magnetic susceptibility at low temperature, showing Curie's law. In any case, the magnetic susceptibility above indicates that the so-called $\gamma$ exponent is 1 ($\chi \propto (T-T_c)^{-\gamma}$) within this mean field theory. As you may have guessed, this is not correct, in general. The correct value is 7/4 in two dimensions, and $\approx 1.24$ in three dimensions. So, the Curie-Weiss susceptibility is *not correct* near $T_c$. In fact, as we mentioned already, $T_c$ itself is not correctly predicted by the mean field theory. However, the Curie-Weiss law above *is* important and useful. This is because the Curie-Weiss susceptibility *is* observed at high temperatures well above the transition temperature. This gives a method to estimate an important microscopic parameter such as $J$.

What is the susceptibility for $T < T_c$? Within this theory, we find that as $h$ is scanned from negative to positive, the magnetization changes from negative to positive abruptly at $h = 0$. So, at precisely $h = 0$, the susceptibility is infinite for any $T < T_c$. At this point, the function $m(h)$ is discontinuous and the derivative is not well-defined.

However, one may proceed with the calculation assuming that $h$ is small and does not cross zero. Thus, we are taking a magnetized system, and considering magnetizing it more, but not less. We are incorporating this fact by assuming that we can get the information on the magnetic susceptibility from Eq. 16.19. In reality, one must recognize the fact that $m$ does *not* change smoothly if $h$ is applied in the opposite direction of $m$. We have a small but finite $m$ and so it is a bad approximation to take the cosh term to be 1. Indeed, if we do that, then we get a negative susceptibility, which does not make any physical sense! So, we need to go to the next order, $\cosh^2\left(\frac{m+h_r}{1+t}\right) \approx \left(1+\frac{1}{2}\left(\frac{m}{1+t}\right)^2\right)^2 \approx 1+m^2$. Thus, multiplying the above equation for $\chi$ by the $\cosh^2$ term, inserting this expression for the $\cosh^2$ term, and noting that $m^2 \approx -3t$ (Eq. 16.17), we get $\left(\frac{t}{1+t}-3t\right)\chi \approx \frac{\mu_B}{Jz(1+t)}$. Thus, we get

$$\chi \;=\; -\frac{1}{2t}\cdot\frac{\mu_B}{Jz} \;=\; \frac{\mu_B}{2k_B}\cdot\frac{1}{T_c-T} \qquad\qquad T<T_c \qquad\qquad (16.23)$$

This means that the "$\gamma'$ exponent" is 1, within the mean field theory of the Ising model.

Note that in order to calculate these critical exponents, it suffices to start from Eq. 16.16 and evaluate $m$ (in the presence of $h_r$) and then calculate $\frac{\partial m}{\partial h}$. Here, we followed a somewhat more general route, with the benefit of learning that the above Curie-Weiss behavior is valid well over $T_c$ within this theory. Indeed, that is where the mean field theory prediction is *correct*, while near $T_c$ the mean field theory generally is not correct. One might already guess that the effect of fluctuations must be important for describing the physical behaviors near the transition, as well as for determining

the transition temperature itself, since it is the fluctuations that we ignored when we employed this mean field approximation.

### 16.1.4 Heat capacity

To calculate the heat capacity, we can use Eq. 16.8 or Eq. 16.15, while keeping in mind that $m = m(T, h)$. Here, we are interested in the near $T_c$ behavior with $h = 0$. In this case, we can take Eq. 16.16 and Eq. 16.17 to express the free energy per spin as

$$g(T) \approx -k_B T \left[ \frac{3t^2}{4} + \log 2 \right] \qquad \text{if } T < T_c, T \approx T_c \tag{16.24}$$

$$g(T) = -k_B T \log 2 \qquad \text{if } T > T_c \tag{16.25}$$

Note that for the last equation, we have extended the region of validity for $T$ to an arbitrarily high temperature, not just near $T_c$, by using Eq. 16.15. Thus, the entropy per spin is given by

$$\frac{S}{N} = -\frac{\partial g}{\partial T} \begin{cases} \approx \frac{3k_B}{2} \cdot \frac{T - T_c}{T_c} + k_B \log 2 & \text{if } T < T_c, T \approx T_c \\ = k_B \log 2 & \text{if } T > T_c \end{cases} \tag{16.26}$$

As expected, we get $k_B \log 2$ for the high temperature regime. The heat capacity per spin is

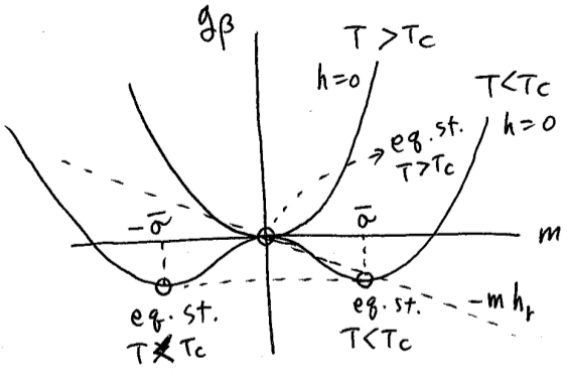
$$\frac{C}{N} = \frac{T}{N} \frac{\partial S}{\partial T} \begin{cases} \approx \frac{3}{2} k_B & \text{if } T < T_c,\ T \approx T_c \\ = 0 & \text{if } T > T_c \end{cases} \tag{16.27}$$

Therefore, there is a jump in the heat capacity. Note on the other hand that the entropy is continuous. Thus, we have a *second order* phase transition. However, the singularity of the heat capacity is more severe in experiment or in a more sophisticated calculation. In the critical phenomena lingo, one defines the "$\alpha$ exponent" as $C \propto (T_c - T)^{-\alpha}$ below $T_c$ and $\propto (T - T_c)^{-\alpha'}$ above $T_c$. So, within the mean field theory of the Ising model, we have $\alpha = \alpha' = 0$. This happens to be sort of[5] correct for the Ising model in two dimensions. However, the correct value for the Ising model in three dimensions is $\approx 0.11$.

[5] The correct solution agrees in that there is no power singularity, but disagrees in that it diverges at $T_c$ logarithmically.

## 16.2 Spontaneous Symmetry Breaking

From the above considerations, we have seen that the system has available two ground states $m = \pm\bar{\sigma}$ for $T < T_c$. This situation is summarized in the diagram below. With $h = 0$ and $T < T_c$, the two minima define two possible equilibrium states. Note that any small field $h$ will skew the potential towards one of the two minima.




Let us ask some basic questions. How are these two states (with $m = \pm\bar{\sigma}$) related to each other? If a system somehow forms in one state, can it evolve to the other state? The answer to the first question is that these two states are related to each other by the symmetry of the underlying Hamiltonian (they are degenerate). The answer to the second question is that they cannot spontaneously transition to the other state.

This means that one of the two states will be a macroscopic state that will be realized.[6] Now, one can note that the physical state is then in a *lower* symmetry state than the Hamiltonian. The Hamiltonian (Eq. 15.12) is invariant under the reversal of all spin directions ($\sigma_i \to -\sigma_i$). Clearly, a state in which all spins are pointing along one direction is *not* symmetric upon the reversal of all spin directions.

Then, we conclude that the condensed state of a ferromagnet is a spontaneously formed state that has a lower symmetry than the Hamiltonian. This is the very important concept of the ***spontaneous symmetry breaking***.

Here, "spontaneous" means ~~that~~ without any external help. Namely, if you just cool down a ferromagnet it will be magnetized along a certain direction according to our theory. For an Ising magnet that we have been considering, it is not possible to

[6] Of course, a real material can form domains. We are not really considering domain formations here, just yet. But, as you will see, soon, we will have to take them into consideration!

tell which of the two directions that the spins will be aligned to. But, it is certain that the system will choose one direction. Had we considered a Heisenberg ferromagnet, then the choice of the direction is continuous – and thus the magnet has infinite choices for its direction! However, the system will choose one of those states.

The concept of the spontaneous symmetry breaking is very fundamental. If somehow you can imagine that we were ant-like creatures living in space which turns out to be a giant Ising magnet of some sort, then we might feel that our world is somehow not symmetric. Such a magnetization that is *part* of the space that we live in will break the time reversal symmetry and the inversion symmetry (i.e. parity). We might feel that this is very strange. While physical laws must be symmetric, we would have to formulate physical laws in the "magnet" in accordance with these broken symmetries, and so it will look like the laws themselves are not symmetric. This is the analogy to what we know is happening in the weak interaction case, where the parity and the time reversal (and the charge conjugation) symmetries are violated. In essence, one can consider the electro-weak theory as the theory that emerges in a new broken-symmetry phase of vacuum.

We have yet to show that the broken symmetry state is stable. I.e., one state does not evolve to another state spontaneously. But before we do the math, let us consider physics.

Since we are considering a crude model of a ferromagnet, we can actually think about a real ferromagnet. A real magnet is likely a Heisenberg magnet than an Ising magnet, and so you may feel that infinite choices of continuous values for the direction mean that a magnet can "slowly transition" so that its magnetization direction points slightly differently. If such a process was possible, then the magnetization direction will slowly evolve to some completely different direction. But, we *know* that this does *not* happen spontaneously in a real magnet. Namely, when one has in hand a magnet, then that magnet will stay in a magnetized state for a long time, unless something dramatic is done to it. By something dramatic, I mean something like a temperature cycling *up through* the Curie temperature and then down, or an application of magnetic field. This type of dramatic disturbance can change the magnetization direction. However, we expect a magnet to stay magnetized in one direction, if we keep it at one temperature without any disturbance. Otherwise, a magnet would be completely useless as a compass!

Why is it then that a spontaneously broken symmetry is stable? It is in fact due to the same reason why the Poincaré period is very large. Suppose that a spin has a probability $p$ to change its direction to a specific direction. In an Ising magnet, this number will be given by $\exp(-\beta O(J))$, since the spin has to flip in an Ising magnet. In a Heisenberg magnet, there would not a large activation energy required, since a spin can cant just a little to change its direction. However, one can see that the

direction of canting can be chosen to be any direction. So, the probability of $p$ for one certain direction will be very small. No matter what the value of $p$ is however, what is important is that it is *not* 1. Now, the probability for all the spins to align along the same direction is given by $p^N$, where $N$ is the number of spins. For any number $0 \leq p < 1$, $p^N \to 0$ for $N \to \infty$, so there is zero chance that a spontaneously broken symmetry state will transition to another state.

This does not mean that system does not remember the symmetry it broke. If the broken symmetry is continuous, as in the Heisenberg magnet case, the following happens. While it is impossible to make all spins suddenly point to another direction, a small local disturbance of canted spins can and do happen. This type of disturbance can propagate as a wave. If the canting of spins is very small on local scales, then the restoring force required vanishes, assuming that spin-spin interaction is not long-ranged. This means that $\omega(\vec{k}) \to 0$ for $\vec{k} \to 0$. This wave is called "magnon" or "spin wave." The magnon is an example of a Goldstone boson. Our discussion here amounts to the Goldstone theorem that states that a broken continuous symmetry implies the emergence of a boson mode, whose energy goes to zero at long wave length. You may remember that we already saw an example of a Goldstone boson – the sound wave, i.e. the acoustic phonon, in a crystal (Section 12.1.4).

## 16.3 Order parameter and the Landau theory

The quantity $m$ (or $M$) is an example of an *order parameter*, a parameter that turns on below the transition temperature and marks the *lower symmetry* in the condensed state. So, it is a parameter that signals a spontaneous symmetry breaking. For an Heisenberg magnet, the order parameter would be vectorial, $\vec{m}$. For a superfluid or a superconductor, the order parameter is a wave function normalized to the condensate density, $\psi(x)$ (cf. Lecture 15). For a gas-liquid transition, the order parameter is the density. For a ferroelectric transition, the order parameter is $\vec{P}$, the polarization.

Landau proposed a simple phenomenological theory of phase transition, which has more than proved its value over time. The Landau theory starts out as a mean field theory of the kind that we have been discussing now. Its focus is symmetry and behaviors near the transition. It assumes that the free energy is analytic near the transition point, just as we found above (as well as in the case of van der Waals theory of gas-liquid transition – which should not surprise you at this point now that we have established the similarity between the two problems). Here, the free energy that we are considering is in fact the "would-be" free energy that we like to minimize with respect to a variational parameter, the "displacement variable" $m$, for a given "force variable" $h$. So, the so-called Landau free energy is a generalization of the would be

grand potential that we considered in Eq. 9.24 and the would be Gibbs free energy that we considered above in Eq. 16.8. By "analytic," we mean that these would be free energies are analytic. We have already seen that non-analytic results (e.g., jump in density in the case of gas-liquid transition and jump in magnetization in the case of a magnetic transition) are obtained from these analytic free energies. Having said this, the analytic property of the would be free energy is actually the property of the mean field theory, and is the limitation of the theory, if it turns out that fluctuations are important. Even in the latter case, however, a generalized Landau theory is very useful.[7]

In any case, we begin with the Landau theory in its simplest form. In fact, we have already kind of covered it above! For a symmetric system ($h = 0$) with a real scalar order parameter $m$, the Landau free energy is defined as

$$G(m, h = 0, T) \;=\; a(T) + \frac{1}{2}b(T)m^2 + \frac{1}{4}c(T)m^4 + ... \tag{16.28}$$

Note that $m$ is "merely" a variation parameter here (while $h = 0$ is the independent variable for describing the physical free energy $G$), but it is conventional to write $G$ as $G(m, T)$. As mentioned above, $G$ can be different types of free-energy/ potential, since in this general theory we do not specify what $h$ and $m$ actually are. $m$ can be any real scalar order parameter, not just the magnetization, and $h$ is its conjugate field. And of course, the theory can be generalized to vectorial or complex order parameters.

For a second order phase transition, we require that

$$b(T) \;=\; b_0\,(T - T_c) \qquad\qquad b_0 > 0 \tag{16.29}$$

$$c(T) \;>\; 0 \tag{16.30}$$

in accordance with the spontaneous symmetry breaking. Keep in mind that $T \approx T_c$ and $m$ is small, by the assumption of this theory. Including the field, we get

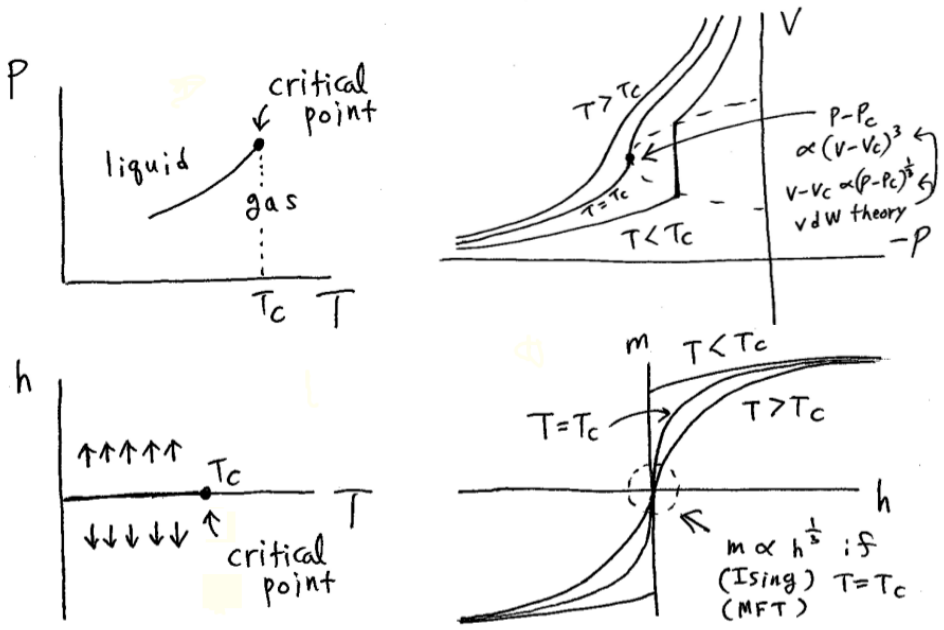
$$G(m, h, T) \;=\; -hm + a(T) + \frac{b_0}{2}(T - T_c)m^2 + \frac{1}{4}c(T)m^4 + ... \tag{16.31}$$

Repeating what we did above, we can calculate various thermodynamic functions near $T_c$, as function of $h, a, b_0$ and $c$. This is left for your work.

## 16.4 Phase diagrams

As a summary, the following diagrams might be helpful.

[7] The Landau theory applied to the superconductivity problem is known as the Ginzburg-Landau theory. When generalized to be of use within the Wilson renormalization group theory, the full theory is referred to as the Ginzburg-Landau-Wilson theory or the Landau-Ginzburg-Wilson theory.




These diagrams compare the gas-liquid phase transition case with the magnetic transition that we have been discussing here using the Ising model. Note the similarities and some differences. One of the main differences is the symmetry in the latter problem with respect to flipping spins, which causes the phase coexistence line to be flat in the $h$-$T$ diagram and the isotherms to have a fixed point at origin in the $m$-$h$ diagram.

In the diagram, it is indicated that the equation of state at $T = T_c$ has the same behavior in the two systems: $h \propto m^3$ and $P_r - 1 \propto (v_r - 1)^3$ (from HW 4.2(a)). This is not surprising at all, since we have already shown that the gas-liquid transition can be described by the Ising model, and both the van der Waals theory and the Landau theory are mean field theories. In the critical phenomena lingo, this means that "$\delta$ exponent" is 3 for this class of models. The correct exponent for the Ising model is 15 for two dimensions and $\approx$ 4.789 for three dimensions.

Lastly, note that this diagram does identify the Curie temperature as the critical temperature. A second order phase transition occurs at that point. On the other hand, if $T < T_c$ and $h$ is swept, we get a first order transition, where the "volume" ($= m$) changes abruptly. Note, however, that in this case, there is *no* latent heat, as can be seen from the symmetry and also from that fact that the phase coexistence line is flat (cf. Clausius-Clapeyron equation).