Notes for Lecture 6

Kinetic theory of gases, cont.

We are studying some classical mechanics as applied to (semi-)classical statistical mechanics. We studied pretty important things – the Poisson bracket, the Liouville's theorem, the definition of an ensemble average, and the basic assumption for the equilibrium state.

Here, we shall discuss an issue that is not totally understood theoretically, yet. It is the issue of the "irreversibility," as the second law of thermodynamics implies. We start by considering a very general theorem that has been invoked in the discussion of the irreversibility.

6.1 Poincaré recurrence

For a Hamiltonian system, one can prove that any bound motion is "almost periodic." Below, we will state what this means in classical mechanics and quantum mechanics, and prove this statement, which goes by the name of Poincaré recurrence theorem.

Historically, this theorem was used against the Boltzmann argument for the irreversibility. However, in fact, this theorem must be viewed as supporting Boltzmann's argument or the general inevitability of the emergence of the irreversibility, if anything.

This consideration actually sheds light on how an irreversibility can come into play, as we start from perfectly reversible problem. In the following, we assume a Hamiltonian that is time-invariant and time-reversal-invariant. That is, the Hamiltonian is not explicitly time dependent, and $H(\vec{q},\vec{p}) = H(\vec{q},-\vec{p})$.

6.1.1 Poincaré recurrence in classical mechanics

Consider a bound motion. Here, by bound motion, we mean a motion bound both in $\vec{p}$ and in $\vec{x}$. Besides, we do not place any restriction on the degrees of freedom – and so the motion can involve any many particles as it is realistic to do so. Poincaré recurrence theorem states that after sufficient time, the system comes back to the original point, or infinitesimally close to it, in phase space.

PROOF ✎ Let us suppose that this is not true. Then, there must be a point, z_0, in phase space that never returns however long a time passes. Here, a point in phase space $z \equiv \{\vec{q}_1, \vec{q}_2, ..., \vec{q}_N, \vec{p}_1, ..., \vec{p}_N\}$, where N is the number of particles involved. Assume that $z(t=0) \equiv z_0$. Since we assumed that z_0 is never reached thereafter, it follows that after an arbitrary finite time τ, z_τ must be at a finite distance away from z_0. Furthermore, it must be possible to enclose a finite volume v_0 around z_0 such that the trajectory will never cross its interior for any $t \geq \tau$. Note that, by definition, v_0 cannot enclose the point z_τ.

Now, let us consider the volume v_0 as an initial phase space volume at $t = 0$. At time τ, this volume will have mapped to another volume v_τ. Now, consider the total volume V_0 that is covered by the time evolution of the initial volume v_0 at $t = 0$. I.e., V_0 is the total volume that is swept by all the phase space trajectories as $t \to \infty$ starting from all the points of the volume v_0 as separate initial conditions. Similarly, we consider the total volume V_τ that is covered by the time evolution of the phase space volume v_τ at $t = \tau$. Note that V_τ and V_0 are both finite, due to our assumption of the boundedness of the system. Furthermore, since v_τ is the time evolution of v_0, we have $V_0 \supset V_\tau$, when these volumes are considered as subsets of the phase space. An important point to keep in mind is that V_τ cannot include the set v_0, by our first assumption in this proof, while V_0 does include the set v_0, by definition.

Next, we in turn treat the volume V_0 as a set of initial conditions, and consider its time evolution. Moreover, at any time, the set of evolved points cannot contain points that are not already contained in V_0. This is clear, since if any such spurious point existed, then that point is clearly a time-evolution of a point from V_0. By the time-invariance of the Hamiltonian, one can see that such a point can be considered as a result of time-evolution of a point from v_0. That is, it must belong in V_0 by definition. In addition, by Liouville theorem, at any time of the evolution, the volume must be equal to the initial volume. To satisfy these two conditions, we see that as a function of time, V_0 is ‘‘almost invariant’’ -- at any given time, the set of points evolved from the set V_0 can be different from from V_0 by a set of points of measure zero, at the most. The same argument holds for V_τ as well.

Now, consider evolving V_τ backwards to time 0. Such a backward evolution must cover the set v_0, by definition. By the time-reversal invariance, we also

know that V_τ must be almost invariant upon this backward evolution as well. The only way that V_τ remains almost invariant and include v_0 is that $V_\tau = V_0$ up to a set of measure zero. But this leads to a contradiction. V_τ cannot contain v_0! QED. ✐

6.1.2 Poincaré recurrence in quantum mechanics

In the quantum mechanical version, the recurrence theorem can be stated as follows. While we are not doing the quantum statistical mechanics just yet, the following consideration turns out to be important when one considers the physics of the Poincaré recurrence in a *classical* system as well. So, we spend some time to prove the following theorem.

As above, consider a time-invariant Hamiltonian system. The wave function of the system for a bound motion can be made as close to the initial wave function as any precision required at a later time. In other words, for any $\epsilon > 0$, it is possible to find T_ϵ such that

$$|\Psi(t + T_\epsilon) - \Psi(t)| < \epsilon \tag{6.1}$$

where Ψ is the normalized wave function of the system. This is the quantum mechanical definition of an "almost periodic" state.

PROOF ✎ The normalized wave function of the system can be written as

$$|\Psi(t)\rangle \;=\; \sum_{j=0}^{\infty} C_j \exp(-iE_j t/\hbar)\,|\psi_j\rangle \tag{6.2}$$

where $|\psi_i\rangle$ is the i-th eigenstate of H. That we have a discrete sum here is due to the assumption of a bound state. These eigenstates form an ortho-normal set of states.
First, note that

$$|\Psi(t + T_\epsilon) - \Psi(t)\rangle \;=\; -2i \sum_{j=0}^{\infty} C_j e^{-iE_j(t+T_\epsilon/2)/\hbar} \sin\left[E_j T_\epsilon/(2\hbar)\right] |\psi_j\rangle$$

$$|\Psi(t + T_\epsilon) - \Psi(t)|^2 \;=\; 4 \sum_{j=0}^{\infty} |C_j|^2 \sin^2\left[E_j T_\epsilon/(2\hbar)\right]$$

As this infinite sum converges (the difference of two finite vectors is finite), it follows that we can truncate the sum to include the sum only

up to a large integer $j = N_\epsilon$ so that the truncation error for the sum is less than $\epsilon^2/2$. Then, the proof would be complete if we showed that

$$4 \sum_{j=0}^{N_\epsilon - 1} |C_j|^2 \sin^2 [E_j T_\epsilon/(2\hbar)] < \epsilon^2/2 \tag{6.3}$$

with a suitable choice of T_ϵ.
Consider adding two periodic functions, characterized by periods T_1 and T_2 respectively. The question is whether the sum of the two periodic functions is also periodic. Clearly, if the two periods are commensurate, $T_1 N_1 = T_2 N_2$ (for some positive integers N_1 and N_2), then the sum is periodic with $N_1 T_1 = N_2 T_2$ as the new period. However, even if T_1 and T_2 are incommensurate, we can approximate the ratio T_1/T_2 as a rational number to any precision required. This fact can be repeatedly used when more than 2, but finite, periods are involved. Thus, one can find a very large period $T_\epsilon \approx N_j T_j = N_j 2\pi\hbar/E_j$ for any $j = 0, ..., N_\epsilon - 1$ to any required precision.
So, we have $E_j T_\epsilon/\hbar = 2\pi N_j + \delta_j$ where N_j is an integer and δ_j is some prescribed small real number, for any $j = 0, ..., N_\epsilon - 1$. The smaller δ_j we require, the larger T_ϵ we must choose, and subsequently the larger N_j we will get. Since we can meet this requirement for any precision for δ_j, we can require

$$|\delta_j| < \frac{\epsilon}{\sqrt{2}} \tag{6.4}$$

for any $j = 0, ..., N_\epsilon - 1$.
Then, $4|C_j|^2 \sin^2 [E_j T_\epsilon/(2\hbar)] = 4|C_j|^2 \sin^2 [\pi N_j + \delta_j/2] = 4|C_j|^2 \sin^2(\delta_j/2) \leq |C_j|^2 \delta_j^2 < |C_j|^2 \epsilon^2/2$. By summing over j, while noting that $\sum_{j=0}^{N_\epsilon} |C_j|^2 \leq \sum_{j=0}^{\infty} |C_j|^2 = 1$, we can see that Eq. 6.3 is satisfied by our choice of T_ϵ. QED.

6.1.3 Poincaré cycle

At first, the Poincaré recurrence theorem, as proved above, may sound like it precludes the emergence of the irreversibility. Indeed, this theorem was used to disqualify the work by Boltzmann, whose theory as we shall discuss below strictly respects the irreversibility.

The funny thing is that the Poincaré recurrence theorem can be proven rigorously as shown above, while Boltzmann's theory cannot be proven rigorously. Nevertheless, Boltzmann's theory is still living and the Poincaré recurrence argument that was used to attack Boltzmann is long disqualified. Mindless math does not necessarily shed light on physics.

So, one should imbue some life to the above math with some physical consideration. If we do so, we come to deal with the matter of the magnitude of the recurrence time scale. The period of the motion that is (nearly) periodic in the Poincaré recurrence theorem is called the *Poincaré cycle*.

What is the value of the Poincaré cycle? It is very large! It is one of those unfathomably large numbers. The point is that it is so large, exponentially larger than the age of the Universe, that it actually supports the emergence of the irreversibility, if anything.

How would one go about estimating the Poincaré cycle? This is an interesting question, which involves some math and physics. You will be able to make a competent estimate in a homework problem. Also, you will see that there is an interesting connection between the very large of the Poincaré cycle and the very short time scale of the irreversibility.

6.2 Boltzmann equation

6.2.1 Simple form

The Boltzmann equation is written as

$$\frac{df_1}{dt} = \left.\frac{\partial f_1}{\partial t}\right|_{collision} \tag{6.5}$$

In our textbook, the right hand side is expressed as a total derivative.

$$\frac{df_1}{dt} = \left.\frac{df_1}{dt}\right|_{collision} \tag{6.6}$$

As the right hand side is not really a differential calculus at all, either of these two notations is actually fine, and we will follow the textbook notation. But, you should keep in mind that the first notation is the one that you will likely to encounter often.

Here,

$$f_1 = f_1(\vec{p}, \vec{q}, t) \tag{6.7}$$

is the **single particle, or one-particle, density function**. It is the density of the particle in the phase space $(\vec{p}, \vec{q})$ with a normalization such that $\int d^3\vec{p}d^3\vec{q}\, f_1 = N$, where N is the number of particles as in the last lecture.

The meaning of the above equation is the following. The left hand side is the total derivative in the multi-variable calculus sense. The right hand side is the change of f_1 due to particle collisions. So, the equation can be re-written as

$$\dot{\vec{p}} \cdot \frac{\partial f_1}{\partial \vec{p}} + \dot{\vec{q}} \cdot \frac{\partial f_1}{\partial \vec{q}} + \frac{\partial f_1}{\partial t} \;=\; \left.\frac{df_1}{dt}\right|_{collision} \tag{6.8}$$

6.2.2 Density functions, distributions, correlations

Here is how f_1 is defined.

$$f_1(\vec{p},\vec{q},t) \equiv \left\langle \sum_{i=1}^{N} \delta(\vec{p}-\vec{p}_i)\delta(\vec{q}-\vec{q}_i) \right\rangle \tag{6.9}$$

is the **single particle, or one-particle, density function**, as it is the ensemble average of the single particle density, as it is written. Note that we are considering a collection of identical particles, and so we are summing over the particle index i – i.e. any particle, if present, will contribute.

$$f_1(\vec{p},\vec{q},t) \;=\; \int d\Gamma \rho(\vec{p}_1,\vec{q}_1,...,\vec{p}_N,\vec{q}_N,t) \sum_{i=1}^{N} \delta(\vec{p}-\vec{p}_i)\delta(\vec{q}-\vec{q}_i) \tag{6.10}$$

$$\;=\; N \int d\Gamma \rho(\vec{p}_1,\vec{q}_1,...,\vec{p}_N,\vec{q}_N,t)\, \delta(\vec{p}-\vec{p}_i)\delta(\vec{q}-\vec{q}_i) \tag{6.11}$$

as the phase-space probability distribution function ρ (Eq. 5.13) must be symmetric with respect to swapping particles for identical particles. Here, $d\Gamma \equiv \prod_i dV_i \equiv \prod_i d^3\vec{p}_i d^3\vec{q}_i$ as in Eqs. 5.11, 5.12. By effecting the integral, we get

$$f_1(\vec{p},\vec{q},t) \;=\; N \int \prod_{i=2}^{N} dV_i\, \rho(\vec{p},\vec{q},\vec{p}_2,\vec{q}_2,...,\vec{p}_N,\vec{q}_N,t) \tag{6.12}$$

$$\;=\; N\rho_1(\vec{p},\vec{q},t) \qquad \rho_1\text{, an unconditional PDF} \tag{6.13}$$

As defined in previous lectures, the unconditional probability distribution function is defined as the one with all other variables integrated out, as here.

Similarly, **many particle density functions** can be defined as well.

$$f_2(\vec{p},\vec{q},\vec{p}\,',\vec{q}\,',t) \equiv \left\langle \sum_{i\neq j} \delta(\vec{p}-\vec{p}_i)\delta(\vec{q}-\vec{q}_i)\delta(\vec{p}\,'-\vec{p}_j)\delta(\vec{q}\,'-\vec{q}_j) \right\rangle \tag{6.14}$$

Again, using the exchange symmetry,

$$f_2(\vec{p},\vec{q},\vec{p}\,',\vec{q}\,',t) = N(N-1)\int\prod_{i=3}^{N} dV_i\,\rho(\vec{p},\vec{q},\vec{p}\,',\vec{q}\,',\vec{p}_3,\vec{q}_3,...,\vec{p}_N,\vec{q}_N,t) \quad (6.15)$$

$$= N(N-1)\rho_2(\vec{p},\vec{q},\vec{p}\,',\vec{q}\,',t) \quad (6.16)$$

where ρ_2 is now the unconditional probability distribution function obtained by integrating out the $6(N-2)$ variables, $\vec{p}_3,\vec{q}_3,...,\vec{p}_N,\vec{q}_N$.

In general, an **s-particle density functions**

$$f_s(\vec{p}_1,\vec{q}_1,...,\vec{p}_s,\vec{q}_s) = \frac{N!}{(N-s)!}\rho_s(\vec{p}_1,\vec{q}_1,...,\vec{p}_s,\vec{q}_s) \quad (6.17)$$

where ρ_s is the unconditional probability distribution function obtained from ρ by integrating out the $6(N-s)$ variables, $\vec{p}_{s+1},\vec{q}_{s+1},...,\vec{p}_N,\vec{q}_N$. Note[1] that $\rho_0 = \rho$ and $\rho_N = 1$.

Note that the **s-particle density function** f_s is directly proportional to the **s-particle distribution function** ρ_s.

The word density in the density function, refers to the density in the phase space. For the single particle density, note that it will become the real space density if $\vec{p}$ is integrated out.

A multi-particle density function will reduce to the product of one particle density functions, if particles are **uncorrelated**. That is, if they are completely independent of one another. This of course is the basic property of the probability distribution function. For *uncorrelated* variables x, y with their separate probability distribution functions ρ_x and ρ_y respectively, the combined probability distribution function is simply $\rho(x,y) = \rho_x(x)\rho_y(y)$. In the current context, $x = (\vec{p}_1,\vec{q}_1)$ and $y = (\vec{p}_2,\vec{q}_2)$ and $\rho_x = \rho_y = \rho_1$, and so we get, for example,

$$\rho_2(\vec{p}_1,\vec{q}_1,\vec{p}_2,\vec{q}_2) = \rho_1(\vec{p}_1,\vec{q}_1)\rho_1(\vec{p}_2,\vec{q}_2) \quad (6.18)$$

$$f_2(\vec{p}_1,\vec{q}_1,\vec{p}_2,\vec{q}_2) = N f_1(\vec{p}_1,\vec{q}_1) f_1(\vec{p}_2,\vec{q}_2) \quad (6.19)$$

if particles are *uncorrelated*.

In other words, the **two particle correlation** is characterized by the difference $\rho_2(\vec{p}_1,\vec{q}_1,\vec{p}_2,\vec{q}_2) - \rho_1(\vec{p}_1,\vec{q}_1)\rho_1(\vec{p}_2,\vec{q}_2)$ or $f_2(\vec{p}_1,\vec{q}_1,\vec{p}_2,\vec{q}_2) - f_1(\vec{p}_1,\vec{q}_1)f_1(\vec{p}_2,\vec{q}_2)$.

[1] There is a typo in the book (page 62) on this point.

6.2.3 Boltzmann equation in full form

Now, let us assume that the total Hamiltonian is given by

$$H(\vec{\mathbf{p}}, \vec{\mathbf{q}}) \;=\; \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + U(\vec{q}_i) \right] + \sum_{i \neq j} V(\vec{q}_i - \vec{q}_j) \tag{6.20}$$

This Hamiltonian contains the one particle Hamiltonian for each particle in a common external potential U and the inter-particle interaction term $\sum V$.

If the interaction is absent, then each particle will be completely independent. And so, there would not be any collisions between them. In that case, the motion of each particle at $\vec{p}, \vec{q}$ will be governed by the single particle Hamiltonian $\frac{p^2}{2m} + U(\vec{q})$. Accordingly $f_1(\vec{p}, \vec{q})$ or $\rho_1(\vec{p}, \vec{q})$ will evolve as this single particle Hamiltonian dictates. This is the precise meaning of the left-hand side of the Boltzmann equation (Eqs. 6.5, 6.6, 6.8). And, so the left hand side of Eq. 6.8 can be re-expressed, using the canonical equation of motion (Eq. 5.15),

$$\vec{F} \cdot \frac{\partial f_1}{\partial \vec{p}} + \frac{\vec{p}}{m} \cdot \frac{\partial f_1}{\partial \vec{q}} + \frac{\partial f_1}{\partial t} \;=\; \left. \frac{df_1}{dt} \right|_{collision} \tag{6.21}$$

where $\vec{F} = -\frac{dU}{d\vec{q}}$. Finally, the collision term is assumed as

$$\left. \frac{df_1(\vec{p}_1, \vec{q}, t)}{dt} \right|_{collision} \;=\; \int d^3\vec{p}_2 d\Omega \left| \frac{d\sigma}{d\Omega} \right| |\vec{v}_1 - \vec{v}_2| \left[f_1(\vec{p}\,'_1) f_1(\vec{p}\,'_2) - f_1(\vec{p}_1) f_1(\vec{p}_2) \right] \tag{6.22}$$

The following comments for this equation is in order.

1. All f_1 functions inside the integral have the same position and time variables $(\vec{q}, t)$. They are not written here for brevity.

2. The position variable $\vec{q}$ and the time variable t here are *coarse grained* variables. The spatial coordinate $\vec{q}$ is averaged over a volume whose linear dimension is a few times the range, d, of the inter-particle potential (typically of the order of an Angstrom). The time is also averaged accordingly. Since the collision time scale is $\tau_c \sim v/d$ where v is the typical speed of a particle ($v \sim 10^2$ m/s), we get $\tau_c \sim 10^{-12}$ sec (i.e. a pico-second). The time is coarse-grained with the resolution of a few times τ_c. Physically, this means that, from the point of view of the Boltzmann equation, which is concerned with the time longer than τ_c and the length longer than d, we may consider the scattering event as happening instantaneously and at a single point in space.

3. The differential crossection ($\left|\frac{d\sigma}{d\Omega}\right|$) and solid angle ($d\Omega = d\phi\, d(\cos\theta)$) are those measured in the center of mass coordinate system for the collision of two particles with initial momentum values $\vec{p}_1$ and $\vec{p}_2$. By definition, $d\Omega \left|\frac{d\sigma}{d\Omega}\right| = d^2\vec{b}$ where $d^2\vec{b}$ means the integration over the area normal to the direction of the impinging particle.

4. As the energy and momentum are conserved in a scattering due to an inter-particle potential $V(\vec{q}_1 - \vec{q}_2)$, the relative speed $|\vec{v}_1 - \vec{v}_2|$ is conserved, a well-known result[2] in mechanics.

5. The momentum values $\vec{p}\,'_1$ and $\vec{p}\,'_2$ are those which, if the time is reversed, will map to $\vec{p}_1$ and $\vec{p}_2$ respectively.

6. So the factor $|f_1(\vec{p}\,'_1)f_1(\vec{p}\,'_2)|$ is proportional to the probability that there will be some particles that will scatter *into* $\vec{p}_1$ and $\vec{p}_2$. This is the *gain* term, from the point of view of $f_1(\vec{p}_1)$.

7. The factor $|f_1(\vec{p}_1)f_1(\vec{p}_2)|$ is proportional to the probability that there will be particles that will scatter *out of* $\vec{p}_1$ and $\vec{p}_2$. This is the *loss* term, from the point of view of $f_1(\vec{p}_1)$.

8. The factor $|\vec{v}_1 - \vec{v}_2|$ is proportional to the incoming flux for the scattering, and the differential crossection describes the scattering probability.

9. The "derivation" for the above equation is given lengthily in the textbook. You may read it, if you are so inclined.

In quantum mechanical language, the collision term can be expressed as

$$\left.\frac{df_1(\vec{p}_1,\vec{q},t)}{dt}\right|_{collision} = \int d^3\vec{p}_2 d^3\vec{p}\,'_1 d^3\vec{p}\,'_2 \delta^4(\vec{P}_i - \vec{P}_f)|T_{if}|^2 \left[f_1(\vec{p}\,'_1)f_1(\vec{p}\,'_2) - f_1(\vec{p}_1)f_1(\vec{p}_2)\right] \tag{6.23}$$

where T_{if} is the T matrix, and the four dimensional delta function means the momentum and energy conservation during the scattering of $\vec{p}_1, \vec{p}_2$ into $\vec{p}\,'_1, \vec{p}\,'_2$ or vice versa.

Without going through the full derivation, you must make sure that the above expressions for the collision term makes sense in terms of the involved elements and also the physical dimension.

[2]This fact can be derived, just by thinking about the physics in the center of mass frame. In the center of mass frame, the momentum is always zero, and so we only need to consider the energy conservation. Since the kinetic energy is proportional to the reduced mass times the square of the relative speed, we see that the relative speed is conserved. So, this is true for any inertial frame. Note that this is valid even if the two particles have different masses.

While the above Boltzmann equation seems to *make sense* considering all of those points listed above, there is one critical *ad-hoc* assumption that is made for the so-called derivation of the collision term. In the derivation, the two particle distribution function $f_2(\vec{p}_1, \vec{p}_2)$ is approximated as $f_1(\vec{p}_1)f_1(\vec{p}_2)$ *before* the collision *but not after* the collision. This assumption, which does not have any support from first principles (yet), is consistent with the fact that particles are more correlated in equilibrium than in non-equilibrium and the fact that collisions establish the equilibrium. However, how to justify this assumption from first principles remains an open task.

6.2.4 Boltzmann H theorem

Assuming that the f_1 function satisfies the Boltzmann equation, one can show that the Boltzmann H function (the negative of the Boltzmann entropy)

$$H(\vec{p}, \vec{q}, t) \;=\; \int d^3\vec{p}d^3\vec{q}\, f_1(\vec{p}, \vec{q}, t) \log f_1(\vec{p}, \vec{q}, t) \tag{6.24}$$

satisfies

$$\frac{dH}{dt} \leq 0 \tag{6.25}$$

The proof was given during class. It must be noted that $\frac{dH}{dt}$ is non-zero only due to the collision term for f_1.

Boltzmann's H theorem is the first case where the irreversibility is derived from microscopic laws that are time-reversal invariant. Here, the words "theorem" and "derived" are a bit too much, if you consider the fact that all of these properties depend on the *ad-hoc* assumption described in the last section.

Nevertheless, one may note that good theorists pay close attention to Nature as well as mathematics. Historically some theorists criticized Boltzmann harshly based on the Poincaré recurrence, which is a rigorous property that seems to directly contradict Boltzmann's theory. One may note that the non-rigorous Boltzmann theory is still widely in use, while the physicists who opposed Boltzmann harshly based on a theory of more mathematical rigor proved rather irrelevant, because that theory does not mean much in relevant time scales.

6.2.5 Perturbative solution

So, we have the Boltzmann's equation. How would one make use of it? We approach solving problems using perturbation[3]. For this approach, one must first get a sense of what perturbation parameter might be.

In item 2 of page 8, we already discussed the fast time scale which is integrated out in Boltzmann equation. What other time scale might be there? An important time scale is the **mean free time** τ_X. This is the mean time between successive collisions. As the particle moves with a speed v, it will scatter with another particle if the volume swept by it $v\tau_X d^2$ has another particle in it. So, we put $v\tau_X d^2\, n = 1$, where n is the density of particles. So,

$$\tau_X \sim \frac{1}{nd^2 v} \sim \frac{\tau_c}{nd^3} \qquad \tau_c \sim d/v \tag{6.26}$$

Typically a gas has a thousand times smaller density than a solid. Noting that the density in a solid is $\sim 1/(10d^3)$, we then can estimate that $\tau_X \sim 10^4 \tau_c$. So, if τ_c is a pico-second, then τ_X is about 10 nano-second.

τ_X is the time scale in which the local equilibrium is established.

Lastly, one may ask what is the time scale involved in the evolution of f_1 due to the "streaming" term df_1/dt? Here, we must note that this streaming term is driven by the external potential. The relevant term is $\dot{\vec{p}} \cdot \frac{\partial f_1}{\partial \vec{p}}$. Typically, the length scale of an external potential is large, say a milli-meter. The inverse time scale that we are interested in is then given by

$$\frac{1}{\tau_U} \sim \dot{p}\frac{\partial}{\partial p} \;=\; -\frac{dU}{dq}\frac{\partial}{\partial p} \sim \frac{v}{l} \tag{6.27}$$

where $l \gtrsim 1$ mm is the length scale over which the external potential U varies significantly[4]. The result is $\tau_U \sim 10$ micro-second or higher.

These considerations suggest the following. Since the collision term is on the order of f_1/τ_X, and since τ_X is the smaller time scale of the two that are relevant for the Boltzmann equation, we can put the equation in this form.

$$\tau_X \left.\frac{df_1}{dt}\right|_{collision} \;=\; \tau_X \frac{df_1}{dt} \tag{6.28}$$

[3]Should you be unfamiliar with the general theory of perturbation, you can look at https://griffin.ucsc.edu/ph105-11/Lecture+#Appendices.

[4]The virial theorem, $\langle T\rangle = \frac{n}{2}\langle U\rangle$, for a power law potential $U \propto r^n$, can be used as a guide to putting $T \sim U$, as done in this estimate.

Here the left hand side is the unperturbed term, having been freed of τ_X. The right hand side is the perturbation term, now proportional to the smallness parameter τ_X/τ_U.

The zero-th order perturbative solution is referred to as the **zero-th order hydrodynamics**, and the first-order perturbative solution is referred to as the **first order hydrodynamics**.

As you can find in the textbook, and as we discussed in class, the zeroth order solution (f_1^0 of Eq. T3.93) describes the local (in time as well as in space) equilibrium solution, where the distribution function is a Maxwell distribution with a local temperature, a local net momentum, and a local density. At the zeroth order, only the local equilibrium is maintained, while there is no heat flow or viscosity that are necessary to even out the inhomogeneous distribution of the temperature and the pressure. These effects come in only in the first order when the streaming term is taken into account.

When the first order solution is sought, there is a widely used approximation – the **single collision time approximation**, or the **relaxation time approximation**. Note first that the first order solution is

$$f_1^1(\vec{p},\vec{q},t) \;=\; f_1^0(\vec{p},\vec{q},t)\,(1+g(\vec{p},\vec{q},t)) \tag{6.29}$$

where g is of the order of τ_X/τ_U. In the relaxation time approximation, one assumes that

$$\left.\frac{df_1}{dt}\right|_{collision} \approx -\frac{f_1^0 g}{\tau_X} \tag{6.30}$$

based on a crude order of magnitude argument. While crude, this approximation is a very practical one to give insight to many difficult real problems. Within this approximation, the first order solution is then given, from Eq. 6.28, by

$$g \;=\; -\tau_X\,\frac{1}{f_1^0}\,\frac{df_1^0}{dt} \tag{6.31}$$

The explicit solution for f_1^1 obtained thus is given in Eq. T3.114.