Notes for Lecture 4

Probability (cont.)

Like in the previous lecture note, let me note couple of things that may be helpful.

4.1 Shifted distribution and cumulant

If a random variable x is shifted, $y = x + a$, then what happens to its cumulants? The answer is that

$$\langle y \rangle_c = \langle x \rangle_c + a \tag{4.1}$$
$$\langle y^n \rangle_c = \langle x^n \rangle_c \qquad \text{for all } n > 1 \tag{4.2}$$

This is easy to figure out. $\tilde{p}_y(k) = \langle \exp(-ik(x+a) \rangle = \exp(-ika)\tilde{p}_x(k)$. Thus, the 2nd characteristic function is given by $\log \tilde{p}_y(k) = -ika + \log \tilde{p}_x(k)$. By definition, $\log \tilde{p}_y(k) = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle y^c \rangle_c$, and similarly for x. We see that the term $-ika$ simply adds to the first order term, but leave all other terms of $\log \tilde{p}_x(k)$ intact.

This fact is useful in understanding the arguments that go into the ***central limit theorem***.

4.2 A correction of the textbook

Eq. T2.51 has a typo and it must read as

$$\lim_{N\to\infty} \log \tilde{p}_y(k) = -a|k|^\alpha \qquad a > 0$$

4.3 Undetermined Lagrange multiplier

Let us look at Eq. T2.73, which reads as

$$S(\alpha, \beta, \{p_i\}) \;=\; -\sum_i p_i \log p_i - \alpha\left(\sum_i p_i - 1\right) - \beta\left(\sum_i p_i F(x_i) - f\right)$$

This is the expression to optimize, when we need to optimize (in this case, maximize) $-\sum_i p_i \log p_i$ given the two constraints $\sum_i p_i = 1$ and $\sum_i p_i F(x_i) = f$. Just like what we encounter in the Lagrangian mechanics, the Lagrange multiplier technique is a mean and clever technique based on the fact that adding zeroes does not do any harm. If we suppose that there are M degrees of freedom ($i = 1, 2, ..., M$), then due to the two constraints we have we got a problem with $M - 2$ independent variables. However, by adding two zero terms above with undermined multipliers, we can simply solve

$$\delta S \;=\; \sum_{i=1}^{M} \frac{\partial S}{\partial p_i} \delta p_i = 0 \tag{4.3}$$

Let us say, for $i = 1$ and $i = 2$ (1,2 are not special, any two indices will do; we pick 1,2 just for the definiteness), we *choose* α and β such that $\frac{\partial S}{\partial p_1} = \frac{\partial S}{\partial p_2} = 0$ which we can do quite freely since α and β are completely arbitrary before we impose these conditions. Then the above equation becomes $\delta S = \sum_{i=3}^{M} \frac{\partial S}{\partial p_i} \delta p_i = 0$. Note that we have exactly $M - 2$ independent variables in this problem, and we can take them to be $p_3, p_4, ..., p_M$. Therefore, for $\delta S = 0$ for any variation, we must conclude $\frac{\partial S}{\partial p_i} = 0$ for $i = 3, .., M$. The net result is that $\frac{\partial S}{\partial p_i} = 0$ for *any* i. So, the mean cleverness of this trick is that we can *view* Eq. 4.3 as applicable for M *independent* variables, with the extra degrees of freedom provided by the included undetermined Lagrange multipliers.

The price is small, compared to the benefit of this trick. In addition to solving the M equations

$$\frac{\partial S}{\partial p_i} \;=\; 0 \tag{4.4}$$

we just have to solve more equations corresponding to the constraints, of which there are two in the current case

$$\sum_i p_i \;=\; 1 \tag{4.5}$$

$$\sum_i p_i F(x_i) \;=\; f \tag{4.6}$$

We got $M + 2$ unknowns to solve, $\{p_i\}$ and α, β, and we got as many equations – so it is doable. (Read page T52. Also, cf. Homework 2.7.)