Due ~~May. 8, Tuesday~~ May. 9, Wednesday

Problem 1 (30 points) *Poincaré period.* This problem will show you how to estimate the Poincaré period.

(a) Consider two oscillatory functions $\psi_1 = \exp(-i\omega_1 t)$ and $\psi_2 = \exp(-i\omega_2 t)$. We assume that $\omega_1 \neq \omega_2$ and furthermore that they are incommensurate with each other. We can interpret ψ_1 and ψ_2 as representing two points on a unit radius circle. As written, at $t = 0$, the two points coincide. After some time the two points will be at a finite angular distance away from each other, since $\omega_1 \neq \omega_2$. The question one can ask is: how long would it take for the two points to come back to the zero angle position within a small tolerance $\Delta\theta$? [Note that $\Delta\theta$ here can identified with $\epsilon/\sqrt{2}$ in Eq. 6.4.] Now, the first point has the period of $T_1 = 2\pi/\omega_1$ and it spends time $t_1 = T_1\Delta\theta/(2\pi)$ in the "sweet zone" ($\Delta\theta$) per revolution. For the second point, $T_2 = 2\pi/\omega_2$ and $t_2 = T_2\Delta\theta/(2\pi)$. We define *coincidence* as the state in which both of these points are in the sweet zone. How frequently this coincidence occurs is characterized by the recurrence period. In order to figure out the recurrence period, it is helpful to consider the following function $t_{12}(\theta_1)$. Without loss of generality, assume $T_1 > T_2$. Consider the moment when point 2 is just entering the sweet zone. Point 1 can be anywhere on the circle at this moment. Let its position be θ_1 ($-\pi < \theta_1 \leq \pi$). As a function of θ_1, we can calculate how long, if at all, the two points will be in the sweet zone together before point 2 leaves the sweet zone after time t_2 has passed. This *duration of coincidence* as a function of θ_1 is defined as $t_{12}(\theta_1)$. Obtain this function.

(b) With the function $t_{12}(\theta_1)$ thus obtained, one can calculate the *average coincidence lifetime* τ_C as

$$\tau_C = \frac{\text{the average of } t_{12}}{\text{the probability of the coincidence}}$$

Note that the probability of the coincidence is proportional to the extent of θ_1 in which $t_{12}(\theta_1)$ is non-zero. τ_C represents the average duration of the coincidence given the condition that a coincidence happened. Show that

$$\frac{1}{\tau_C} = \frac{1}{t_1} + \frac{1}{t_2}$$

(c) Explain the last result by considering the following. Suppose that a coincidence is observed at time t. Consider the probability that this coincidence will be broken within the next infinitesimal time interval dt. By definition, this probability is dt/τ_C. It can also be expressed in terms of t_1, t_2, and dt. Show clearly that this consideration leads exactly to the above result ("Matthiessen's rule").

(d) Now generalize to the case when there are M oscillators with mutually incommensurate periods. By using the same argument, prove that

$$\frac{1}{\tau_C} = \sum_{i=1}^{M} \frac{1}{t_i}$$

where $t_i \equiv T_i \Delta\theta/(2\pi)$ and T_i is the period of the i-th oscillator.

(e) In the limit of running the system to the infinity of time, we can express the probability of coincidence in two ways. The first way is simple: $\left(\frac{\Delta\theta}{2\pi}\right)^M$. The second way is τ_C/T_{pc}, where T_{pc} is the recurrence period, or the Poincaré cycle. By equating these two quantities, and using the results above, show that

$$T_{pc} = \frac{2\pi}{\omega_{ave}} \frac{1}{M} \exp\left[(M-1)\log\frac{2\pi}{\Delta\theta}\right]$$

where ω_{ave} is the average frequency of the M oscillators.

(f) As an example of a classical mechanical system, consider normal modes of linearly coupled oscillators as applicable to the molecular vibrations in a one dimensional solid. The typical frequency is the Debye frequency so that $2\pi/\omega_{ave} \sim 10^{-12}$ sec. If there are 101 atoms in a linear chain, then $M = 100$ (excluding one translational mode that has zero frequency). Evaluate and discuss the value of T_{pc}.

(g) As a quantum mechanical example, we may consider an electron gas. When this system goes out of equilibrium (due to an external field or a temperature gradient), it relaxes to a local equilibrium by collision processes. This time scale is what we discussed in class as τ_X. By a conservative estimate, $\tau_X \sim 1$ ns, while it is typically *much* shorter than 1 ns. Note that τ_X gives a lower bound for the energy width ΔE through the Heisenberg uncertainty relation. To estimate the corresponding lower bound of the number of frequency modes, one must divide ΔE by the energy increment of the system. This is roughly[1] $\hbar^2 4\pi^2/(2mL^2)$ where L is the sample size or the mean free path $v_F\tau_X$, whichever is larger. So, for $\tau_X \sim 1$ ns, we have $L \sim v_F\tau_X \sim 1$ mm at least, if $\hbar v_F \sim 5$ eV Å. Estimate M based on these parameters and the corresponding T_{pc}. Discuss the relationship between the irreversibility time scale (recall that τ_X is the time scale in which the local equilibrium is established) and T_{pc}, if any. [Hint: $\hbar c = 1973$ eV Å, and $mc^2 = 0.511$ MeV would be useful.]

Problem 2 (20 points) *Boltzmann equation and conductivity.* Consider a dilute gas system consisting of mobile carriers of charge e and mass m in a lattice of

[1] This estimate is based on the independent particle picture. In an interacting system, this estimate will tend to become smaller due to the formation of "heavy fermions."

fixed ions. The ions ensure that the system is charge-neutral overall and also provide additional scattering mechanism for the mobile carriers. Recall from class that the zeroth order solution for the Boltzmann equation is given by the local (in time and space) equilibrium Maxwell distributions. If there is no external force, then the system will evolve to the global equilibrium according to the first order hydrodynamics. Here, in this problem, we will assume that this already occurred. Thus, we consider, as the *new* zeroth order solution, the *global equilibrium* Maxwell distribution

$$f_1^0(\vec{p}, \vec{q}, t) \;=\; \frac{n}{(2\pi m k_B T)^{3/2}} \exp\left(-\frac{\vec{p}^2}{2mk_BT}\right)$$

where n and T are constants, in both space and time.

(a) A weak constant electric field $\vec{E}$ is turned on, and leads to a new steady state (cf. Section 1.5). Within the relaxation time (τ_X) approximation (cf. Eq. 6.30), calculate the distribution function f for the steady state, to first order.

(b) From that distribution function, calculate the electric conductivity σ, defined by $\vec{j} \equiv n e \vec{u} = \sigma \vec{E}$, where $\vec{u} \equiv \langle \vec{p} \rangle / m$ is the *drift velocity.*

Problem 3 (20 points) Problem 3.9 of Kardar, *Boltzmann equation and viscosity.*

Problem 4 (20 points) *Ideal gas from a novel approach.*

(a) Divide the gas into many small identical volume elements such that the number of gas molecule in each volume element is $\ll 1$. Consider the statistics of counting particles this way[2] and show that $\Delta N = \sqrt{\langle N^2 \rangle_c} = \sqrt{N}$, where N is the average number of molecules. Clearly state where the ideal gas assumption is used in your reasoning.

(b) Now, using this result, Eq. 7.42, and the discussion following that equation, set up a differential equation for N with respect to the chemical potential μ (T and V are fixed). Show that the result can be written as

$$\left.\frac{\partial N}{\partial \mu}\right|_{V,T} \;=\; \frac{N}{k_B T}$$

(c) Show that this means that μ is a function of the form

$$\mu \;=\; k_B T \log(nA)$$

where $A = A(T)$ is a function of T only and $n = N/V$.

[2]Note that this method naturally invokes a grand canonical ensemble.

(d) Consider the Gibbs free energy, $G = E - TS + PV = F + PV = \mu N$ (Eq. 2.34) and the corresponding thermodynamic identity Eq. 2.18. Show that $Nd\mu = VdP$ when T is held fixed. By plugging this result into the differential equation obtained in part (b), show that the equation of state $PV = Nk_BT$ is derived.

(e) Combine the results of parts (c), (d) and the fact that the partition function is given by (Eq. 7.32)

$$Z = \exp(-\beta F) = (V/\lambda^3)^N/N!$$

to prove that $A(T) = \lambda(T)^3$, where $\lambda(T) = h/\sqrt{2\pi m k_B T}$ is the thermal De Broglie wavelength.

Problem 5 (10 points) *Maxwell-Boltzmann, Fermi-Dirac, and Bose-Einstein.* Show that, using the results of the last problem, the Maxwell distribution (as defined in the phase space, not just in the $\vec{p}$ space; cf. problem 2) can be written as

$$f_1(\vec{p}, \vec{q}, t) = \frac{\exp\left(-\beta\left[\varepsilon(\vec{p}) - \mu\right]\right)}{h^3}$$

where $\varepsilon(\vec{p}) = \frac{p^2}{2m}$. Explain why you would have expected this, based on what you know about (or you can refresh your memory about) the Fermi-Dirac or Bose-Einstein occupation number. You also need to explain the h^3 factor precisely (cf. Section 7.1.1).

Problem 6 (20 points) Problem 4.7 of Kardar, *Microcanonical Ensemble.*

Problem 7 (20 points) Problem 4.9 of Kardar, *Grand Canonical Ensemble.*

Problem 8 (20 points) Problem 4.12 of Kardar, *Canonical Ensemble.*