Due Apr. 24, Tuesday.

Problem 1 (20 points) Using the Sterling's formula (Eq. T2.66) and the Taylor expansion of the logarithm, prove the following.

(a) The binomial distribution (Eq. T2.19) becomes a Gaussian distribution, in the limit of $Np_A \to \infty$, $Np_B \to \infty$. Hint: Write $N_A = Np_A + x$, and note that $|x| \ll Np_A$ when Np_A is large. Take the logarithm of the binomial distribution, obtain the leading order expansion for x.

(b) The Poisson distribution Eq. 3.29 becomes a Gaussian distribution, in the limit of $\lambda = Np \to \infty$. Hint: Write $k = \lambda + x$, and note that $|x| \ll \lambda$ when λ is large. Take the logarithm of the Poisson distribution, obtain the leading order expansion for x.

Problem 2 (20 points) A set of telephone lines is to be installed between two towns A and B. Town A has 3000 telephones. The question is how many telephone lines would be enough to service the residents of town A. Clearly 3000 lines would be an overkill. Let us assume that each telephone gets used to connect to town B for 2 minutes a day, on the average. For simplicity, we will further assume that the times of phone calls are completely random. Find the minimum number of phone lines from A to B to ensure that the connection failure rate from town A to town B is 1 percent at the most. Hint: Math will become simpler if you use the result of the previous problem.

Problem 3 (20 points) Let us consider particle decays like we did in class, but this time, suppose that we are not interested in a time distribution, but a frequency distribution. We shall assume that the measured byproduct of the particle decay is a photon, which is emitted at a characteristic frequency (ω_0 below). So, we are counting photons while measuring its frequency. It is a general result that any such photon (whether it be from a radioactive decay, an atomic decay or a decay in a solid state) obeys the following Lorentzian probability distribution function $p_L(\omega)$.

$$p_L(\omega) = \frac{1}{\pi}\frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2}$$

$$\Gamma = \frac{1}{2\tau} \qquad \tau = 1/e \text{ lifetime (cf. LN 3, pages 4,5)}$$

(a) Show that the above distribution function does satisfy the normalization condition. Show that the mean value of the frequency is ω_0. Hint: A contour integral may be necessary for the first one.

(b) Consider successive many measurements $\omega_1, \omega_2, \omega_3, ..., \omega_N$ of N photons. Like in Section T2.5, consider the variable $\Omega = \sum_i \omega_i$. Assuming that each measurements are uncorrelated, compute, exactly, the probability distribution function of Ω. Hint: Use the characteristic function.

(c) Discuss whether this probability behaves as dictated by the central limit theorem or not. Discuss why or why not.

Problem 4 (10 points) Using the connected cluster analysis (pages T39 and T44; "linked cluster theorem"), expand

(a) $\langle x^5 \rangle$ in terms of the cumulants of the 5th order and lower

(b) $\langle x_1^3 x_2 \rangle$ in terms of cumulants and joint cumulants

Problem 5 (10 points) Prove Eq. T2.42 from Eq. T2.41. Hint: First, define $\vec{y} = \vec{x} - \vec{\lambda}$ and then diagonalize the matrix $\overleftrightarrow{C}$ by an orthogonal transformation (which is possible since $\overleftrightarrow{C}$ is a real symmetric matrix).

Problem 6 (20 points) Problem 2.9 of Kardar. Hint: Think Poisson distribution.

Problem 7 (20 points) Problem 2.12 of Kardar.