Each problem is worth 100 points. Grades for only three problems will be used, and problem 1 will always be included. Please organize your solutions as well as you can. **First, write your name down on your solution sheet.**

Good luck!

Problem 1 Grades for 3 out of 4 following parts will be used.

(a) *Fluctuation-dissipation theorem.* Show either that $C_V = k_B \beta^2 \Delta E^2$ or that $\chi = \frac{\partial M}{\partial h} = \beta \Delta M^2$ using the general property of the (grand or Gibbs) partition function.

$\frac{\partial^n \log Z}{\partial X^n} = \overline{\langle Y^n \rangle}_c$ if $Z = \sum_\alpha e^{XY_\alpha + \text{ other terms (no } Y_\alpha)}$ where α is an appropriate variable (e.g., micro state or energy) that Y is dependent on. $\beta \equiv 1/(k_B T)$. E is the energy. $\Delta E^2 \equiv \overline{\langle (\mathcal{E} - E)^2 \rangle}$ where $E \equiv \overline{\langle \mathcal{E} \rangle}$. C_V is the heat capacity at constant volume. M is the total magnetization/magnetic-moment. $\Delta M^2 \equiv \overline{\langle (\mathcal{M} - M)^2 \rangle}$ where $M \equiv \overline{\langle \mathcal{M} \rangle}$. h is the magnetic field.

(b) Explain what the equipartition theorem is. No derivation is necessary.

(c) "The equipartition theorem remains valuable even for quantum mechanical problems, as it helps a qualitative understanding of the thermal behavior in the quantum regime." True or false? Explain why, very briefly.

(d) *Without* using integrals, preferably, or using absolutely minimum mathematics necessary, in any case, show that the (total) heat capacity $\sim N(T/T_C)^{3/2} k_B$ up to a numerical constant for a He gas (^{4}He) in the Bose-Einstein condensate (BEC) state. N is the total number of He atoms, and T_C is the BEC transition temperature.

Grades for 2 out of 3 following problems will be used.

Problem 2 Consider a He gas with pressure P and in equilibrium with a metal surface. The energy of a He atom on the metal surface with no kinetic energy is $-\phi$. He atoms can freely move around on the metal surface, forming a two dimensional gas system. Let us call this phase the "surface gas" phase as opposed to the "volume gas" phase outside of the metal surface. We do not consider interactions between He atoms. Find the number of He atoms per unit area for the surface gas, as a function of P, T, ϕ and fundamental constants

(a) for T near room temperature, and

(b) for T below the Bose-Einstein condensate transition temperature for the volume gas.

In both cases, assume that P is near atmospheric pressure.

[Notes: For full credit, you need to show the derivation for key quantities such as μ, for part (a), using the classical ideal gas model. For part (b), the known value of μ can be used without derivation. Helpful formula: $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$.]

Problem 3 Consider the *transverse* Ising model

$$H = -J \sum_{\langle i,j \rangle} \sigma_{z,i}\sigma_{z,j} - h\mu_B \sum_i \sigma_{x,i}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $J > 0$ and $\langle i, j \rangle$ means the nearest-neighbor pairs (assume z such pairs for a given spin). By employing the mean field theory, find

(a) expressions for $m_z = \overline{\langle \sigma_{z,i} \rangle}$ and $m_x = \overline{\langle \sigma_{x,i} \rangle}$,

(b) expression for the critical temperature (T_c) for $h = 0$, and

(c) the magnetic susceptibility $\chi_x = \frac{\partial m_x}{\partial h}$ and $\chi_z = \frac{\partial m_z}{\partial h}$, assuming small h.

[Notes: (1) For part (c), you might (roughly) guess your answers first, and then try to derive them. (2) Useful formula: $\tanh \delta \approx \delta - \frac{1}{3}\delta^3$.]

Problem 4 A classical ideal gas is contained in a cylinder with a freely sliding piston lid. The cylinder is mounted in the upright position. The lid has mass M and surface area A. Its variable vertical position is y, measured from the bottom of the container. We consider the classical regime only. Ignore gravitational force for gas molecules. However, the lid experiences the gravitational force.

(a) Find the partition function Z for the combined system (gas plus lid), in terms of the above parameters, and $\lambda = h/\sqrt{2\pi m k_B T}$ for the gas and other fundamental constants.

(b) Consider the gas part of the quantity $-k_B T \log Z$. What free energy is it equal to: the Helmholtz free energy, or the Gibbs free energy, of the gas? Is it consistent with the basic principle of the statistical mechanics? For full credit, you need to demonstrate the equality in detail. Just stating the principle would give you partial credit only.

[Notes: (1) All key quantities, such as the partition function, must be derived, for full credit. (2) Helpful formulae: $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$, and $N! \approx N^N e^{-N}$ if $N \gg 1$.]