Lecture 19

Thursday, March 15, 2012

Superconductivity

A perfect conductor is not necessarily a superconductor

Historically, the Dutch physicist Heike Kamerlingh Onnes discovered the superconductivity in Mercury in 1911. At about 4 K, the resistivity dipped down to a practically zero value. Later experiments confirm that the current in a superconductor is maintained for a long time (> 1 year), showing that the superconductivity does entail the infinite conductivity.

How to describe a material with an infinite conductivity?

Let us recall the equation of motion for the electron gas, $m\frac{d\vec{v}}{dt} + m\frac{\vec{v}}{\tau} = -e\vec{E}$, where $\vec{v}$ is the drift velocity. Normally, the steady state solution to this equation would lead to Ohm's law. However, the case of infinite conductivity requires some care. For infinite conductivity, $\tau \to \infty$, and thus the 2nd term on the LHS can be ignored. What we have then is

$$m\frac{d\vec{v}}{dt} = -e\vec{E}$$

In terms of the current density $\vec{j} = -ne\vec{v}$, we have

$$\frac{d\vec{j}}{dt} = \frac{ne^2}{m}\vec{E}$$

This equation definitely explains the "perfect conductor" aspect of the superconductor, since $\vec{E} = 0$ allows a solution for a finite $\vec{j}$. However, London and London (1935) pointed out that this equation is not fundamental for superconductors.
We consider the case when there is a vector potential, but no scalar potential. Then, we have $\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$. So, what we have is (replacing the total time derivative to partial for $\vec{j}$ also):

$$\frac{\partial\vec{j}}{\partial t} = -\frac{ne^2}{mc}\frac{\partial\vec{A}}{\partial t}$$

Using the Maxwell equation $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}$ (assuming no displacement current) this equation means $\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{4\pi n e^2}{mc^2}\frac{\partial \vec{A}}{\partial t}$ which on taking the curl on both sides becomes $\vec{\nabla} \times \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{4\pi n e^2}{mc^2}\frac{\partial \vec{B}}{\partial t}$. Using the vector identity $\vec{\nabla} \times \vec{\nabla} \times \vec{F} = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$, and using $\vec{\nabla} \cdot \vec{B} = 0$, we get

$$\nabla^2 \left(\frac{\partial \vec{B}}{\partial t}\right) = \lambda_L^{-2} \left(\frac{\partial \vec{B}}{\partial t}\right)$$

where

$$\lambda_L \equiv \sqrt{\frac{mc^2}{4\pi n e^2}}$$

This means that for a perfect conductor, $\frac{\partial \vec{B}}{\partial t}$ has an exponential dependence over the distance. For instance, in one dimension, $\frac{\partial \vec{B}}{\partial t} \propto \exp(-x/\lambda_L)$ if the sample exists in the $x > 0$ region. [The other solution that has ($\partial \vec{B}/\partial t \propto \exp(x/\lambda_L)$ will not be consistent with the solution with zero (or exponentially small) $\vec{E}$ inside the sample.]

Then, we will conclude that for the interior of the sample, whose distance from the sample surface is much greater than λ_L, $\frac{\partial \vec{B}}{\partial t} = 0$. I.e., $\vec{B}$ is a time independent constant.

The odd thing is that for a superconducting state, that time independent constant is always zero, signifying that there is some deeper principle that just the infinite conductivity. For instance, imagine a material which can be cooled down to a certain temperature to make its conductivity go to infinity. Suppose that this was "just a perfect conductor" for which $\tau \to \infty$ as T goes down. Imagine that initially the conductor is in a finite conductivity phase (normal phase) and there is a $\vec{B}$ field inside the conductor. As the temperature is lowered, let us assume that the $\vec{B}$ field does not change. [Most metals are weak paramagnets with a temperature independent susceptibility, as we shall see later, so this assumption is a very good one.] So, $\frac{\partial \vec{B}}{\partial t} = 0$ even in the normal phase. What would happen if the temperature reaches the perfect conducting temperature? In this "perfect conductor" model, there is no reason why $\frac{\partial \vec{B}}{\partial t} = 0$ would break down in any step, and so the initial $\vec{B}$ field would remain the same. This is NOT what happens in a superconductor, though.

Meissner Effect

Meissner and Ochsenfeld (1933) discovered that, as opposed to the discussion just made, the $\vec{B}$ field is completely expelled from the sample as the transition temperature T_c is reached from above.

The meaning of the Meissner effect is that **superconductors are perfect diamagnets**. Namely, for a long thin sample (to avoid discussions about geometry dependent demagnetizing field) and an applied field B_a we have $B = B_a + 4\pi M = 0$ or $M = -B_a/4\pi$.

Note that a perfect diamagnet is not necessarily a superconductor, nor is a perfect conductor. However, a superconductor is both a perfect diamagnet and a perfect conductor.

Critical Field

It is observed that if a strong enough field is applied, then the superconductor turns into a normal metal. The minimum applied field B_a is called the critical field B_{ac}.

The work done on a superconductor is $dF_s = -\vec{M} \cdot d\vec{B}_a$. ($\vec{H} = \vec{B}_a$ inside the superconductor.) With $\vec{M} = -\frac{1}{4\pi}\vec{B}_a$ (Meissner field for thin long sample), we have $dF_s = \frac{1}{4\pi}B_a dB_a$. Thus,

$$F_s(B_a) - F_s(0) = \frac{B_a^2}{8\pi}$$

In the presence of the field, the energy of the superconductor goes up. Eventually, when the energy becomes high enough so that $F_s(B_a) = F_N(B_a) \approx F_N(0)$, where F_N is the (very weakly B_a dependent) normal state free energy, then $B_a = B_{ac}$. That is, we have

$$F_N(0) = F_s(B_{ac}) = F_s(0) + \frac{B_{ac}^2}{8\pi}$$

London Equation

London and London (1935) proposed that $\nabla^2\left(\frac{\partial \vec{B}}{\partial t}\right) = \lambda_L^{-2}\left(\frac{\partial \vec{B}}{\partial t}\right)$ be replaced by

$$\nabla^2 \vec{B} = \lambda_L^{-2} \vec{B}$$

since this theory is similar to the above theory based on the "perfect conductor" but does not have the un-observed solution for superconductors where $\vec{B} \neq 0$ inside the superconductor. This revolutionary proposal was an important step in the theory of superconductors.

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n e^2}}$$

is the celebrated London penetration depth, which is the same as defined above, but now the meaning of the subscript L is proper. λ_L is typically on the order of 100 to 1000 Å.

Note that the above equation for $\vec{B}$ is equivalent to converting the "perfect conductor" equation $\frac{\partial \vec{j}}{\partial t} = -\frac{ne^2}{mc}\frac{\partial \vec{A}}{\partial t} = -\frac{c}{4\pi}\frac{1}{\lambda^2}\frac{\partial \vec{A}}{\partial t}$ to

$$\boxed{\vec{j} = -\frac{c}{4\pi}\frac{1}{\lambda_L^2}\vec{A}}$$

This is the **London equation**. This is what London brothers proposed to replace the usual Ohm's law for a superconductor. The ultimate theory by Bardeen, Cooper, and Schrieffer (1957) derives this result from their microscopic theory (**BCS theory**).

The London equation might seem very strange, since it equates the vector potential and $\vec{j}$, which is an *observable*. How can this be? If you remember from E&M (and quantum mechanics) the principle of "gauge invariance," then you know that one can transform $\vec{A} \to \vec{A} + \vec{\nabla}\chi$ and at the same transform $\phi \to \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$ and all is well, since observable quantities ($\vec{E}, \vec{B}$) remain unchanged. The time-tested London equation means that in a superconductor this **gauge invariance is broken**! As a crystal is a broken symmetry state (translation, rotation), a superconductor is also a broken symmetry state! Namely, **a superconducting state is a broken gauge symmetry state**. This has some notable consequences. For instance, there is a phonon like particle that emerges (Goldstone boson; Josephson plasmon). Another consequence is of course the Meissner effect, which can be viewed as a total exclusion of photons below a certain frequency. This can be summarized as the photon dispersion going from the normal dispersion $\epsilon_k = \hbar c k$ to an anomalous one $\epsilon_k \approx \Delta_{cutoff} + \frac{\hbar^2 k^2}{2m}$ (with $mc^2 = \Delta_{cutoff}$) inside a superconductor. I.e., a superconductor is a state in which photons become massive! This realization (by Anderson) strongly motivated the theory of Higgs boson and the theory of mass generation in matter, a hot topic nowadays in high energy physics.

Pippard non-local electrodynamics

It turns out that the London theory was not enough to explain all superconductors. In some superconductors, it turns out that the London equation needs to be modified as

$$\vec{j}(\vec{r}) = -\frac{c}{4\pi}\frac{1}{\lambda_L^2}\int \overleftrightarrow{M}(\vec{r}-\vec{r}')\vec{A}(\vec{r}')d\vec{r}'$$

as suggested by Pippard (1953). Here, $\overleftrightarrow{M}(\vec{r}-\vec{r}')$ is a matrix function $\propto \exp(-|\vec{r}-\vec{r}'|/\xi)$ has the length scale of ξ, which is the **coherence length**. [This type of non-local electrodynamics has a precedence in Chambers' non-local Ohms law $\vec{j}(\vec{r}) = \int \overleftrightarrow{M}'(\vec{r}-\vec{r}')\vec{E}(\vec{r}')d\vec{r}'$ where $\overleftrightarrow{M}'(\vec{r}-\vec{r}')$ is a function $\propto \exp(-|\vec{r}-\vec{r}'|/l)$ where l is the mean free path of the electron.]

So, what is this coherence length ξ? It is the length scale over which the super-current $\vec{j}$ will not change much, in a spatially varying magnetic field. As such it is the length scale associated with the basic quantity (Cooper pair) of the superconductor.

Pippard argued that, for the transition that sets in at T_c (superconducting transition temperature) only those electrons at energy $\sim k_B T_c$ from the Fermi energy will contribute. Then, the uncertainty in the wave vector of a superconducting wave function must be $\Delta k \lesssim k_B T_c/\hbar v_F$, which leads to the size of the superconducting wave function $\Delta x \gtrsim \frac{1}{\Delta k} \sim \frac{\hbar v_F}{k_B T_c}$. This length scale is called ξ_0.

The BCS theory confirms Pippard's theory. The first step of the BCS theory is two electrons forming a "Cooper pair," a bound state formed by two electrons through an effective attractive interaction. The binding energy of these two electrons is, not surprisingly, on the order of $k_B T_c$. A Cooper pair is formed by linear combinations of pair states with two electrons at $\vec{k},\uparrow$ and $-\vec{k},\downarrow$. So, a particular pair state formed by $\vec{k},\uparrow$ and $-\vec{k},\downarrow$ has the momentum $\hbar\vec{k}$ in the relative coordinate system, and an s-wave state is formed by summing over all $\vec{k}$

values. In order to figure out the rough minimum size of the Cooper pair wave function, let us think how one can give enough energy to a particular pair state involving $\vec{k},\uparrow$ and $-\vec{k},\downarrow$ by introducing the uncertainty Δk in $k = |\vec{k}|$. Clearly if $\hbar v_F \Delta k \gtrsim k_B T_c$, we would have given enough uncertainty in energy to break the pair. Thus, the stability of the Cooper pair demands that $\hbar v_F \Delta k \lesssim k_B T_c$, which is the same as above condition presented by Pippard, leading to $\xi_0 \sim \frac{\hbar v_F}{k_B Tc}$. In the standard (i.e. weak coupling) BCS theory

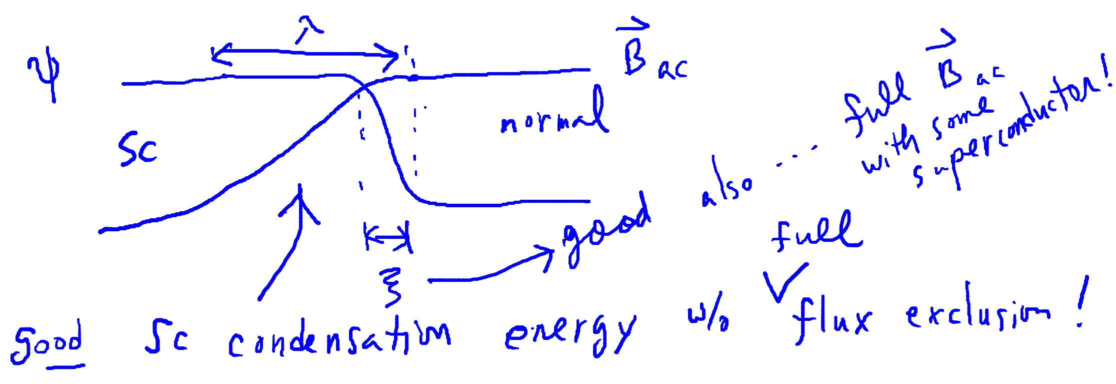
$$\xi_0 = \frac{\hbar v_F}{1.76\pi k_B T_c}$$

ξ_0 (intrinsic coherence length) is the Pippard coherence length ξ for a pure sample at $T = 0$ for which the mean free path $l \to \infty$. As l becomes small, it makes ξ decrease accordingly. As the material becomes impure, l becomes small, and it makes ξ decrease accordingly ($\xi \le \xi_0$). In contrast, the penetration depth of the $\vec{B}$ field increases ($\lambda \ge \lambda_L$). The ratio λ/ξ is denoted by κ.

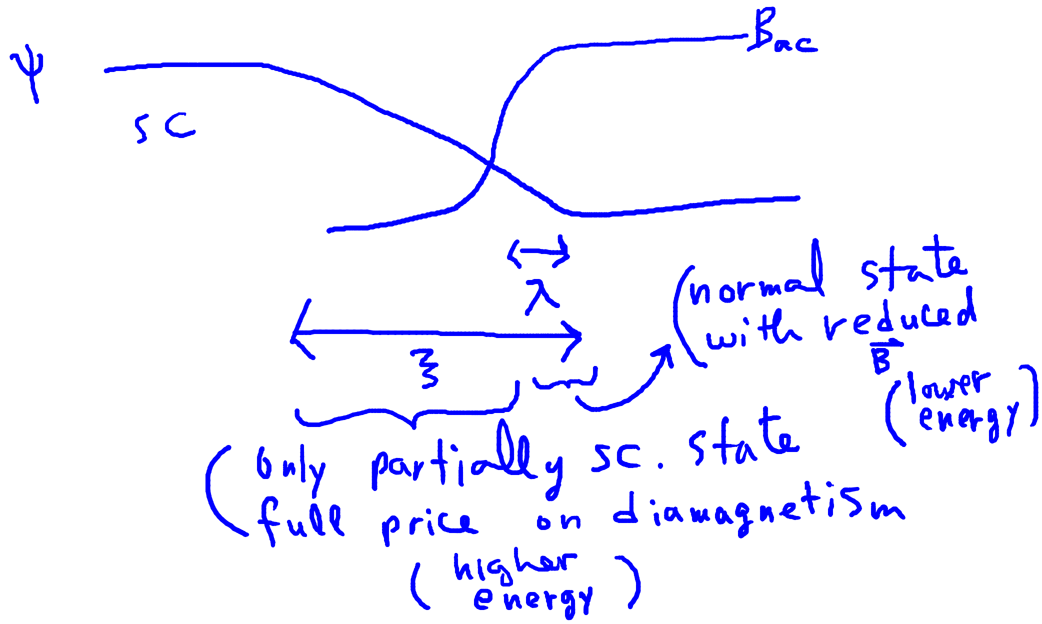
ξ_0 is typically on the order of 1000 Å.

Type I and Type II superconductors

What happens when $\kappa = \frac{\lambda}{\xi} \gg 1$? Consider a critical field B_{ac}, and a possible domain formation. On one side, there is a superconductor, and on the other side there is a normal metal. In the domain boundary, the full superconducting pairs and the partial magnetic field coexists in the length scale of $\lambda - \xi$. This is a happy situation where the superconducting condensate does not have to pay the full price of the diamagnetism. So, in this case the domain energy is negative -- the domain formation is preferred! This is the case of type II superconductors.



Now, let us consider the opposite case, $\kappa = \frac{\lambda}{\xi} \ll 1$. In this case, the domain boundary consists of mainly partially superconducting state, while paying the full price of the diamagnetic energy. This means positive domain energy, and this means that the domain formation is discouraged! This is the case of type I superconductors.



Type II superconductors are important, since they have two critical fields, B_{c1} and B_{c2}. Above B_{c1}, magnetic fields are admitted partially, and they form flux lattices. Above B_{c2}, the superconductivity finally disappears. B_{c2} is much greater than B_c of type I superconductors, which is a good thing, since the critical current is related to the critical field that it generates.

Superconducting order parameter

London (1954) is credited with the first assertion that the superconducting wave function is that of a single quantum state occupied by all superconducting charge carriers. This is the so-called the superconducting order parameter. In a steady state:

$$\psi(\vec{r}) = \psi_0 e^{i\theta(\vec{r})}$$

$$\psi_0 = \sqrt{n_s}$$

The London equation is derived as follows from this important point of view.

$$\vec{p}\,\psi = -i\hbar\vec{\nabla}\psi = \hbar\vec{\nabla}\theta\,\psi$$

$$p\psi = -i\hbar\nabla\psi - \hbar\nabla\theta\,\psi$$

$$\vec{p} = 2m\vec{v}_s - \frac{2e}{c}\vec{A} \leftarrow \text{(Cooper pair)}$$

Uniform state $\vec{p} = 0$

Thus, $\vec{j}_s = -n_s e \vec{v}_s = -\frac{n_s e^2}{mc}\vec{A}$

The Ginzburg Landau theory (1950) developed a full phenomenology using this concept of the quantum wave function for the whole system, and continues to be one of the most valued theories in physics.

Within this view, the superconductivity amounts to a rigid many wave function, which is attributed to the energy gap and the many body nature of the wave function.

$$\vec{j}_0 = \frac{\hbar q}{2im}\left(\psi_0^* \vec{\nabla}\psi_0 - (\vec{\nabla}\psi_0)^*\psi_0\right) \qquad \text{QM, e.g. Sakurai}$$

$$\vec{j} = \vec{j}_0(\psi_0 \to \psi) - \frac{q^2\vec{A}}{mc}|\psi|^2 \qquad H = \frac{1}{2m}\left(\vec{p} - \frac{q\vec{A}}{c}\right)^2 + \text{pot.}$$

$\psi \approx \psi_0$ (energy gap, many-body coherence)

$$\vec{j}_0 = 0 \qquad \vec{j} = -\frac{q^2\vec{A}}{mc}n_s$$

London Equation

pair density $\to \frac{n}{2}$ $\quad q = -2e, \; m \approx 2m_e$

So all factor of 2's cancel $\to$ agree with the above!

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

$$-\vec{\nabla}^2\vec{A} = \frac{4\pi}{c}\vec{j} = -\frac{4\pi\, q^2 n_s}{mc^2}\vec{A} \qquad \lambda \sim \text{a few } 100\ \text{Å}$$

$$\vec{\nabla}^2\vec{A} = \frac{\vec{A}}{\lambda^2} \qquad \lambda = \sqrt{\frac{mc^2}{4\pi q^2 n_s}}$$

$\vec{B}$ field is screened within the length scale λ

Meissner Effect

Steady state $\quad$ No electrostat. pot.

Infinite Conductivity

$$\frac{\partial \vec{j}}{\partial t} = 0 \qquad \vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \qquad \vec{E} = 0$$

BCS Theory

The BCS theory (1957) suggested that the two electrons form a bound state by the electron-lattice interaction. This was motivated by the isotope effect $T_c \propto M^{-\alpha}$, where α is often, but not always, about 0.5, and M is the isotope mass of the ion. How can this be? Electrons repel each other greatly, of course, by the Coulomb interaction, but on a slow time scale, they do feel the retarded effect of the electron-lattice interaction. Often this story is told. Imagine a bed where a spring is very slow to respond. Let us imagine two people who share the bed, but on a different time schedule. They avoid each other, like two electrons repelling each other. The first person uses the bed and goes out to do something else. The second person comes and uses the bed. Imagine that the bed is so slow to respond that the imprint made by the first person is still not gone. From the second person's point of view, in this situation, the energy would be lower if that person can fit snug into that imprint. This is how phonons mediate an effective attractive electron-electron interaction, even if the Coulomb interaction is very large.

The formation of Cooper pairs is necessary but not sufficient condition. Cooper pairs are bosons, and they Bose-condense, i.e. they occupy the same quantum state. It is important to note that the phases of Cooper pairs are coherent, like photons in a laser light.

Josephson Tunneling

The striking consequence of the Cooper pair and the phase coherence is the Josephson current. A bold proposal, initially regarded with some skepticism, this effect is now the basis of very fine measurements of magnetic flux, and also used as a standard for voltage. Kittel p. 289-293 is a good reading on this.

The DC Josephson effect means that a DC current will flow between two superconductors connected by a thin insulator. $J = J_0 \sin(\theta_2 - \theta_1)$.

The AC Josephson effect means that an AC current will flow if a voltage is applied across the junction. $J = J_0 \sin\left[\theta_2(0) - \theta_1(0) - \frac{2eVt}{\hbar}\right]$.