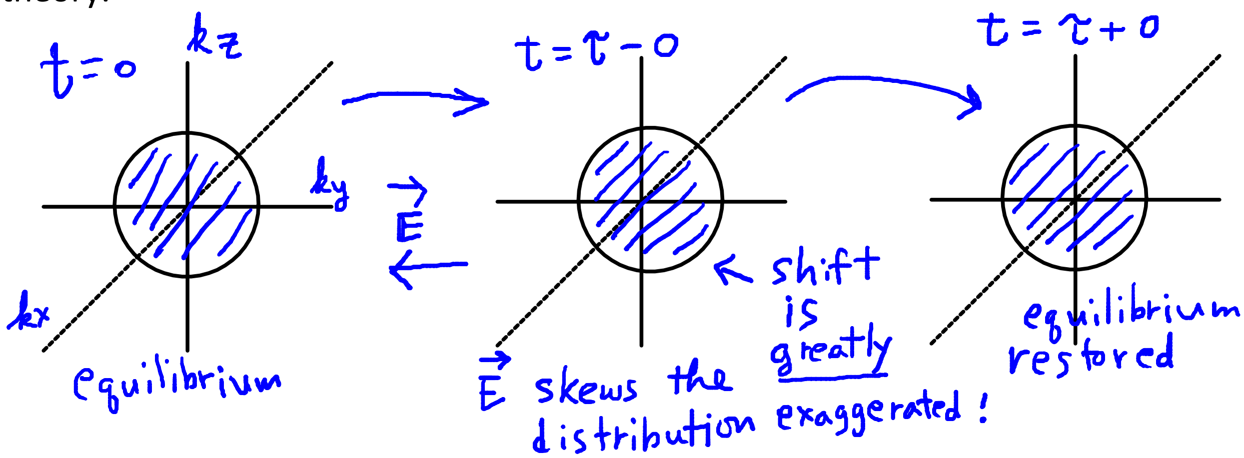
Lecture 12

Thursday, February 16, 2012

Conduction of electricity

The free electron theory explains why metals conduct electricity. Here is a simple theory.



In this picture, one imagines that the electron system is in an equilibrium, and is driven out of the equilibrium due to the electric field. On a time scale of τ, **the relaxation time**, the system goes through collisions and brought back to the equilibrium state again, the history of the past completely forgotten and the whole process starting all over again.

The relaxation time τ is **the average time** for the electron to travel before it is scattered, in a simple picture. [But, note that the normal scattering due to electron-electron interaction does not count. Just like in the phonon problem, only the Umklapp scattering counts as far as the electron-electron interaction is concerned. So, the relaxation time is generally longer than the electron scattering time.]

How much current does the $\vec{E}$ field drive?

As we shall see, the relation $\hbar\dot{\vec{k}} = \vec{F}$ (a semi-classical equation of motion) is valid for free electrons or electrons in a crystalline potential, with $\hbar\vec{k}$ being the **crystal momentum** in general. We can use this equation for each electron state, with $\vec{F} = -e\vec{E}$. But what we ultimately want is the average over all electrons:

$$\frac{1}{N_e}\sum\hbar\dot{\vec{k}} = \frac{1}{N_e}\sum\vec{F}$$

The sum is over all occupied states. The RHS is simply $\vec{F}$, as $\vec{F}$ is a constant. Thus, this equation means that the average momentum $\frac{\sum \hbar \vec{k}}{N_e}$ increases gradually due to $\vec{F}$. Note that at the equilibrium $\frac{\sum \hbar \vec{k}}{N_e} = 0$, while it will be a certain finite value at $t = \tau - 0$. We call that value $m\,\vec{v}_d$, where $\vec{v}_d$ is the **drift velocity**. Note that the drift velocity has nothing to do with the individual velocity of each electron, which is on the order of the Fermi velocity. Instead, $\vec{v}_d$ is the average net velocity gained by the electron over the time scale τ. Thus we get:

$$\frac{m\vec{v}_d}{\tau} = \vec{F} = -e\vec{E}$$

The current density that is gained by the applied field is

$$\vec{j} = -ne\vec{v}_d = -ne\left(-\frac{e\vec{E}\tau}{m}\right) = \frac{ne^2\tau}{m}\,\vec{E}$$

which defines the conductivity

$$\sigma = \frac{ne^2\tau}{m}$$

This is called the **Drude conductivity**.

Typical values

For normal metals, the conductivity is on the order of $(\mu\Omega\ cm)^{-1}$, which is in the cgs unit 1e18 Hz (note that Ω is an SI unit).

With this value of σ, τ can be estimated from $\tau = \frac{m\sigma}{ne^2}$, with $n \approx 0.1\ \text{Å}^{-3}$, as 3e-14 sec. That is 30 femto seconds. For a typical range of conductivity encountered in normal metals, τ ranges from ~ 1 femto second to ~ 10 nano second (at low T, pure metal).

The **mean free path**, $l = v_F\tau$, measures the distance that the electron travels between collisions. This is given by $\sim 10\ \text{Å}$ to 1 mm with a more typical value being between $100\ \text{Å}$ to 1 μm.

Lastly, the drift velocity for a very large current density such as $1000\ \text{A/cm}^2$ is very tiny. $v_d = \frac{j}{ne} \approx 10^{-11}\, c \sim 10^{-9}\, v_F$. This is a truly negligible fraction of v_F, but since $\vec{v}_d$ is all in the same direction, it adds up to a macroscopic effect, while $\vec{v}_F$ adds up to a zero macroscopic effect since it is in all directions.

NOTE: In "strongly correlated electron systems" (notably high temperature superconductors, manganites, and heavy fermions), unusually low conductivity is often observed in metals: $(\mathrm{m\Omega\ cm})^{-1}$ or less. Often this implies a mean free path less than the lattice constant! This is an indication that the "free electron physics" as we know it and as we introduced it here is in a serious trouble.

Thermal conductivity and Wiedemann-Franz law

The thermal conductivity $\kappa = \frac{1}{3}\, v\, l \frac{C}{V}$ (see Lecture 10). In this free electron model, $v = v_F$, $l = v_F \tau$, $C = \frac{\pi^2}{2} N_e k_B \left(\frac{T}{T_F}\right) = \frac{\pi^2}{2}\, N_e k_B^2 \frac{T}{\epsilon_F} = \pi^2 N_e k_B^2 \frac{T}{m v_F^2}$. Thus,

$$\kappa = \frac{1}{3} v_F^2 \tau \pi^2 N_e k_B^2\, \frac{T}{m v_F^2} \frac{1}{V} = \frac{\pi^2}{3} k_B^2 \tau\, n \frac{T}{m}$$

Note that $\frac{n\tau}{m}$ enters κ, as it enters $\sigma = \frac{n\tau e^2}{m}$. So, this suggests that

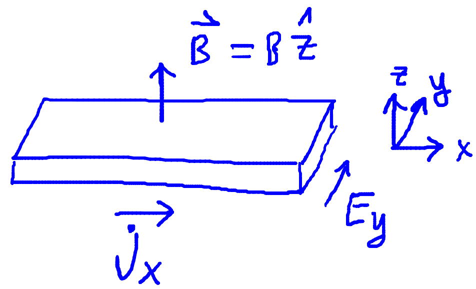
$$\frac{\kappa}{\sigma T} = \frac{\pi^2}{3} \left(\frac{k_B}{e}\right)^2 \equiv \text{Lorenz number}$$

The fact that this number is observed to be universal in many common metals (see table 5 of Kittel) is called **Wiedemann-Franz law**, lends much support to the free electron theory.

In pure metals, the electricity and the heat are both carried by the electron, and this fact underlies the Wiedemann-Franz law. However, note that the assumption of this law is that the same relaxation time τ applies to the electric conductivity and to the thermal conductivity, which was our assumption above. That this is *not* always the case can be realized when we consider the $T \ll \theta_D$ regime where the electron is scattered by a low energy phonon with $k \ll k_D$. Since, $k_D \sim k_F$, this means that the wave vector $\vec{k}$ of the electron hardly changes, meaning that the scattering is a forward scattering. Such forward scattering is not efficient at all in slowing down the electric current, while it is efficient in slowing down the heat current, since the typical energy of the phonon involved in such a scattering is $k_B T$. So, Wiedemann-Franz law is not valid in this regime. **Its ranges of validity are very low temperatures, where the impurity scattering dominates, and high**

temperatures ($T \gg \theta_D$), where short wave length phonons abound. In both cases, the change of the electron wave vector due to collisions is on the order of k_F.

Hall Effect



In this experiment, a B field is applied perpendicular to the sample (a "Hall bar"), say the z axis, while a current is induced by an electric field along the x axis. As the charge carrier experiences the Lorentz force, it will acquire some momentary current along the y direction. In the steady state, the Hall bar will develop a charge accumulation at the edges and the corresponding electric field E_y. The Hall coefficient R_H is defined as

$$R_H = \frac{E_y}{j_x B}$$

Within the free electron model, it is straightforward to calculate R_H. As before, we use the semi-classical equation of motion (which we will prove couple of lectures later): $\vec{F} = \hbar\dot{\vec{k}}$ with $\vec{F} = -e\left(\vec{E} + \frac{1}{c}\vec{v}_{\vec{k}} \times \vec{B}\right)$. Here, $\vec{v}_{\vec{k}}$ is the group velocity at $\vec{k}$, i.e. $\hbar\vec{k}/m$ for the current theory. Defining the drift velocity $\vec{v}$ as before (i.e. as the average of $\vec{v}_{\vec{k}}$ over all electrons) we get

$$m\frac{d\vec{v}}{dt} = -e\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

where $\vec{v}$ is the drift velocity. Note the transition from $\vec{v}_{\vec{k}}$ ($\sim$ Fermi velocity) to $\vec{v}$ (a very tiny fraction of Fermi velocity). As before, we apply the relaxation time approximation, which can be implemented by making the substitution $\frac{d}{dt} \to \frac{d}{dt} + \frac{1}{\tau}$. This substitution can be best understood as considering $\vec{v}$ as being related to the probability that the electron distribution will survive without collision $P(t)$: $P(t)\left(1 - \frac{dt}{\tau}\right) = P(t + dt)$ and so $\frac{dP}{dt} = -\frac{P}{\tau}$. Thus, in the absence of an external force, $\frac{d\vec{v}}{dt} + \frac{\vec{v}}{\tau} = 0$. In the presence of an external force

$$\frac{d\vec{v}}{dt} + \frac{\vec{v}}{\tau} = \frac{\vec{F}}{m}$$

In the current problem, we have

$$\frac{d\vec{v}}{dt} + \frac{\vec{v}}{\tau} = -\frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

We look for a steady state solution ($\frac{d\vec{v}}{dt} = 0$) with $v_y = v_z = 0$, $\vec{B} = B\hat{z}$, and $E_z = 0$.

$$v_x = -\frac{\tau e}{m} E_x$$
$$v_y = 0 = -\frac{\tau e}{m}\ (E_y - \frac{1}{c} v_x B)$$

The first equation is what we already know. $j_x = -nev_x = \sigma E_x$, with $\sigma = \frac{ne^2\tau}{m}$. The second equation means: $\frac{E_y}{v_x B} = \frac{1}{c}$. Thus, $R_H = \frac{E_y}{j_x B} = -\frac{1}{nec}$.

$$R_H = -\frac{1}{nec}$$

This is the famous Hall coefficient (in the SI unit, $R_H = -\frac{1}{ne}$). The simplicity of this equation is the result of the simple model, the free electron model. While this theory is marvelously successful in many ways, it also comes short in important ways. According to this model, note that R_H is always negative. This is not true in general can be seen in Table 4. For instance, the Hall coefficient is positive for Be, Al, In, and As. It is also very large for Sb and Bi. All these anomalies need to be explained by the "band theory" which we will now explore. A more fundamental failure of the free electron theory is the inability to explain why certain substances such as Si, diamond or GaAs become insulators/semiconductors.