Understanding Debye T^3 law

$$C_V \propto T^3 \quad \text{if} \quad T \ll \theta_D$$

How does this come about? By considering the following facts, you should be able to understand how that comes about.

i. $E = E_0 + 3\int_0^{\omega_D} d\omega D(\omega) n(\hbar\omega)\hbar\omega$
   (3 for 3 polarization branches for acoustic phonons)

ii. Change the variable from $\frac{\hbar\omega}{k_B T}$ to x.

iii. Note that at low T, the upper limit of the integral for x goes to ∞.

iv. Notice that the integral is then just a number (if converges) – so look for the temperature factors that came out of the integral in step ii! Then, consider, $C_V = \left(\frac{\partial E}{\partial T}\right)_V$. That is it!

v. As a bonus, you should be able to understand the statement in the next page.

θ_D is the Debye temperature, $n(\hbar\omega)$ is the Bose-Einstein distribution function, and $D(\omega)$ is the DOS. Often times, $\hbar$ in $n(\hbar\omega)$ may be omitted (like in a following page!). $n(\hbar\omega) = \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}$

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$$C_V \sim N_l \left(\frac{T}{\theta_D}\right)^3 k_B \quad \text{if} \quad T \ll \theta_D$$

N_l is the number of lattice points, θ_D is the Debye temperature, and $\sim$ means "up to a dimensionless numerical factor".

Understanding Debye T^3 law

The same can be deduced from the following more physical reasoning.

i. At a given temperature, some phonons are excited in large numbers ("active"), while some photons are not excited at all ("inactive").

ii. The value $\omega_T \sim k_B T/\hbar$ is the crossover phonon frequency scale. That is, if ω is too large (several times ω_T or greater), then that phonon is *exponentially suppressed* (show it) and there is practically zero such phonons excited. These are inactive phonons.

iii. On the other hand, if ω is small compared to ω_T (say a quarter of it or less), it is a good approximation to say $n(\omega) = \frac{1}{e^{\omega/\omega_T}-1} \approx \frac{1}{1+\frac{\omega}{\omega_T}-1} = \frac{\omega_T}{\omega}$. I.e., the occupation number $n(\omega)$ is high, and these phonons are active. The energy of each active phonon mode $\sim$ [   ].

iv. How many phonon modes are active? All phonon modes with $\omega \lesssim \omega_T$. Note that all phonon modes with $\omega \lesssim \omega_D$ account for the total number of acoustic phonon modes possible (N_l). Find the wave vectors (k_T and k_D) corresponding to ω_T and ω_D and consider the $\vec{k}$ space volume for $|\vec{k}| < k_T$ and $|\vec{k}| < k_D$. Taking the ratio of the two volumes , the number of active phonon modes = [   ].

v. Due to the exponential suppression of high frequency phonons, it is clear that they can be ignored. Then, the total energy of the system is

$E \sim$ Answer of iii $\times$ Answer of iv $=$ [   ].

vi. Take the temperature derivative and you should get the answer of the previous page!

The answer in each blank should contain numbers and (a subset of) these symbols only, θ_D, k_B, T, N_l.

Going beyond Debye T^3 law

The absolute beauty of the Debye law is that it does not depend on any fact other than the fact that the translational symmetry of space is broken by the solid state. The resulting Goldstone bosons (phonons) are responsible for the Debye law.

One may wonder whether there is any other continuous symmetry that can be broken? Indeed. For instance, consider EuO. It is a semiconductor. Furthermore, it is a ferromagnet (below $\sim$ 69 K). How does it become a magnet? Just like any other rare earth containing substance does: each rare earth site (Eu site for EuO) carries a magnetic moment, which *aligns in one direction*. So, what kind of continuous symmetry does EuO break?

In any case, Goldstone bosons in this case are called **magnons** (or spin waves). A ferromagnetic spin wave behaves as $\omega \propto k^2$ for low lying modes.

Which of the previous steps (either in page 3 or in page 1) need modification? [Hint: just one step needs modification.]

What power law T will C_V show at low temperatures?

Will this magnon C_V dominate over the phonon C_V at low temperatures?