Appendix C -- Field Momentum

Friday, March 09, 2012

Why is it that the canonical momentum for the position vector $\vec{r}$ becomes $m\vec{v} + \frac{q}{c}\vec{A}$ when there is an electromagnetic vector potential $\vec{A}$? For a formal discussion, I'd refer you to field theory books such as Landau and Lifshiftz, "The classical theory of fields, page 45." However, in this note, what I like to do is two things. First, I will show an intuitive example in which we can see why the canonical momentum must be $m\vec{v} + \frac{q}{c}\vec{A}$. Second, I will show explicitly that if the non-relativistic Hamiltonian is written correctly with the above fact then the correct EM force equation results.

(1) For the first part, we consider the case of no electrostatic potential. Here is the example problem.

Consider an infinitely long solenoid with current I and N turns per unit length. The finite size of the wire can be ignored.

(a) Calculate the magnetic induction **B**, and the vector potential **A** outside and inside the solenoid. In particular, show that the vector potential **A** outside the solenoid cannot be zero.

(b) Consider a particle at rest with charge q, placed at position **R** outside the solenoid. Suppose the solenoid current goes through a sudden change δI in a short time δt, e.g. due to heating or cooling. Calculate the velocity acquired by the particle from this change. Identify the conserved total momentum in this process.

Here is the solution to this problem.

Field momentum

(a) Using Stoke's theorem to $\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$ and the cylindrical symmetry, it is readily seen that the $\mathbf{B}$ field outside the solenoid is 0. Inside the solenoid, Stoke's theorem gives $\mathbf{B} = \frac{4\pi}{c} NI\hat{z} = B_0\hat{z}$. Applying Stoke's theorem to $\mathbf{B} = \nabla \times \mathbf{A}$ on a circle of a constant radius ρ from the center of the solenoid, and noting the cylindrical symmetry, it is readily shown that $\mathbf{A}$ should be non-zero outside the solenoid, as it should at least have an azimuthal component A_ϕ that should satisfy $A_\phi 2\pi\rho = B_0\pi a^2$. In fact this gives the solution $\mathbf{A} = B_0\frac{\pi a^2}{2\pi\rho}\hat{\phi} = B_0\frac{a^2}{2\rho}\hat{\phi}$. Similarly, inside the solenoid, $\mathbf{A} = B_0\frac{\pi\rho^2}{2\pi\rho}\hat{\phi} = B_0\frac{\rho}{2}\hat{\phi}$. (Standard cylindrical coordinates (ρ, ϕ, z) are used here.)

(b) Corresponding to a change of current δI, there is a change of the vector potential$\delta\mathbf{A} = \delta B_0\frac{R}{2}\hat{\phi}$. Thus, there is an electric field $\mathbf{E} = -\frac{1}{c}\frac{\delta\mathbf{A}}{\delta\mathbf{t}}$ during time δt. The impact on the charged particle is then $q\mathbf{E}\delta t = -q\delta\mathbf{A}/c$. The corresponding gain in velocity is, $\delta\mathbf{v} = -q\delta\mathbf{A}/(mc)$. This "transfer of momentum" suggests that $m\mathbf{v} + \frac{q}{c}\mathbf{A}(\mathbf{R})$ is the total momentum that is conserved, where the 2nd term $\frac{q}{c}\mathbf{A}(\mathbf{R})$ can be identified as the field momentum.

(2) For the second part, the Hamiltonian is written as (see e.g. Landau and Lifshiftz, "The classical theory of fields," page 46).

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

$$\frac{\partial H}{\partial \vec{p}} = \dot{\vec{r}}$$

$$\vec{p} - \frac{q}{c}\vec{A} = m\dot{\vec{r}} = m\vec{v}$$

$$\frac{\partial H}{\partial \vec{r}} = -\dot{\vec{p}}$$

$$\nabla(\vec{\alpha}\cdot\vec{\alpha}) = 2\vec{\alpha}\times(\nabla\times\vec{\alpha}) + 2(\vec{\alpha}\cdot\nabla)\vec{\alpha}$$

$$\dot{\vec{p}} = \frac{q}{c}\vec{v}\times\vec{B} + \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)\cdot\frac{q}{c}\nabla\vec{A} - q\nabla\phi$$

$$= \frac{q}{c}\vec{v}\times\vec{B} + \frac{q}{c}\vec{v}\cdot\nabla\vec{A} - q\nabla\phi$$

$$\dot{\vec{p}} = m\dot{\vec{v}} + \frac{q}{c}\dot{\vec{A}} = \frac{q}{c}\vec{v}\times\vec{B} + \frac{q}{c}\vec{v}\ \cdot\nabla\vec{A} - q\nabla\phi$$

$$= \frac{q}{c}\vec{v}\times\vec{B} + \frac{q}{c}\left(\dot{\vec{A}} - \frac{\partial\vec{A}}{\partial t}\right) - q\nabla\phi$$

The equation of motion:

$$m\dot{\vec{v}} = \frac{q}{c}\vec{v}\times\vec{B} + \frac{q}{c}\vec{E}$$

which is, of course, the correct equation of motion.