Homework 6

Due 2/23/2012

1. [15 points] (a) Show that the kinetic energy of a free electron gas at absolute zero temperature is given by $E_0 = \frac{3}{5}N\epsilon_F$. (b) Find the expression for the pressure $P = -\partial E/\partial V$ and the bulk modulus $B = -V\partial P/\partial V$, at $T = 0$, both in terms of N, V and ϵ_F. (c) Estimate the contribution of the conduction electrons to B for potassium and compare your answer to the experimentally measured bulk modulus $0.37 \times 10^{10}\ \mathrm{Nm^{-2}}$. You can use the data given in table 1 of page 139 (see a copy below).

2. [30 points] Consider valence electrons in a metallic material. We keep the spirit of the free electron approximation, but employ a slightly more general approach, by assuming a density of states function $D(\epsilon)$ valid for $\epsilon \approx \epsilon_F$ without specifying what the function really is except that $D(\epsilon_F)$ is a finite number. Our only other assumption is that $T \ll T_F$.
   a. The conservation of the electron number N means

   $$\int_0^{\epsilon_F} D(\epsilon)d\epsilon = \int_0^{\infty} D(\epsilon)f(\epsilon,\mu,T)d\epsilon = N$$

   Note that in this problem the Fermi Dirac function $f(\epsilon, T)$ is treated as a function of *three independent variables* $f(\epsilon,\mu,T)$, purely for a mathematical reason. Physically, $\mu = \mu(T)$ for fixed N and V, and so μ and T cannot be independent of each other.

   Show that this equation can be re-written as

   $$\int_{\epsilon_F}^{\mu} D(\epsilon)d\epsilon = \int_0^{\infty} D(\epsilon)\big(f(\epsilon,\mu,T=0) - f(\epsilon,\mu,T)\big)d\epsilon$$

   Note that $f(\epsilon,\mu,T=0)$ is a purely mathematical construct where $\mu = \mu(T)$, while $T = 0$. So, it is a step function like the true Fermi Dirac function at $T = 0$, but the step is not at $\mu(T=0) = \epsilon_F$ but at $\mu(T)$.

   b. Show that the function $f(\epsilon,\mu,T=0) - f(\epsilon,\mu,T)$ is an odd function of $\epsilon - \mu$. Also, show that it is exponentially small when $|\epsilon - \mu| \gg k_B T$, meaning that it is appreciably different from 0 only when $|\epsilon - \mu| \lesssim k_B T$.

c. Now, assume that $\mu \approx \epsilon_F$ at all temperatures of interest ($\ll T_F$). Using the results of a and b, and the Taylor expansion $D(\epsilon) \approx D(\mu) + D'(\mu)(\epsilon - \mu)$, show that, to leading order,

$$\mu \approx \epsilon_F - \frac{\pi^2}{6}\frac{D'(\epsilon_F)}{D(\epsilon_F)}(k_B T)^2$$

Use the "method of iteration" (If you are not familiar with the method, see https://griffin.ucsc.edu/ph105-11/Lecture%2B?action=AttachFile&do=view&target=A01-Perturbation.pdf). You can use $\int_0^\infty dx \frac{x}{e^x+1} = \frac{\pi^2}{12}$ without proof.

d. Consider the case when $D(\epsilon) \propto \epsilon^\alpha$. Show that

$$\mu \approx \epsilon_F \left(1 - \alpha \frac{\pi^2}{6}\left(\frac{T}{T_F}\right)^2\right)$$

e. What is the value of α for the free electron dispersion in 1 dimension, 2 dimensions, and 3 dimensions? Find the finite temperature correction to μ, in each case, by plugging in the value of α.

f. Employing the same techniques (i.e. using $f(\epsilon, \mu, T) = \{f(\epsilon, \mu, T) - f(\epsilon, \mu, T = 0)\} + f(\epsilon, \mu, T = 0)$, $D(\epsilon) \approx D(\mu) + D'(\mu)(\epsilon - \mu)$, and the method of iteration), and using the above result $\mu \approx \epsilon_F - \frac{\pi^2}{6}\frac{D'(\epsilon_F)}{D(\epsilon_F)}(k_B T)^2$, show that the total energy is given by

$$E = \int_0^\infty d\epsilon \epsilon D(\epsilon) f(\epsilon, \mu, T) \approx E(T = 0) + \frac{\pi^2}{6}(k_B T)^2 D(\epsilon_F)$$

Now, consider the case of free electrons in 3 dimensions. Using the fact that $E(T = 0) = \frac{3}{5}N\epsilon_F$ (problem 1) and $D(\epsilon_F) = \frac{3N}{2\epsilon_F}$ (lecture), show that

$$E \approx \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4}N\epsilon_F\left(\frac{T}{T_F}\right)^2$$

and that

$$C_V \approx \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F}\right)$$

3. [15 points] Kittel 6.4 (see below).

4. [10 points] Kittel 6.5 (see below).

5. [10 points] Kittel 6.6 (see below).

Table 1 Calculated free electron Fermi surface parameters for metals at room temperature

(Except for Na, K, Rb, Cs at 5 K and Li at 78 K)

Valency	Metal	Electron concentration, in cm^{-3}	Radius[a] parameter r_s	Fermi wavevector, in cm^{-1}	Fermi velocity, in $cm\ s^{-1}$	Fermi energy, in eV	Fermi temperature $T_F = \epsilon_F/k_B$, in deg K
1	Li	4.70×10^{22}	3.25	1.11×10^8	1.29×10^8	4.72	5.48×10^4
	Na	2.65	3.93	0.92	1.07	3.23	3.75
	K	1.40	4.86	0.75	0.86	2.12	2.46
	Rb	1.15	5.20	0.70	0.81	1.85	2.15
	Cs	0.91	5.63	0.64	0.75	1.58	1.83
	Cu	8.45	2.67	1.36	1.57	7.00	8.12
	Ag	5.85	3.02	1.20	1.39	5.48	6.36
	Au	5.90	3.01	1.20	1.39	5.51	6.39
2	Be	24.2	1.88	1.93	2.23	14.14	16.41
	Mg	8.60	2.65	1.37	1.58	7.13	8.27
	Ca	4.60	3.27	1.11	1.28	4.68	5.43
	Sr	3.56	3.56	1.02	1.18	3.95	4.58
	Ba	3.20	3.69	0.98	1.13	3.65	4.24
	Zn	13.10	2.31	1.57	1.82	9.39	10.90
	Cd	9.28	2.59	1.40	1.62	7.46	8.66
3	Al	18.06	2.07	1.75	2.02	11.63	13.49
	Ga	15.30	2.19	1.65	1.91	10.35	12.01
	In	11.49	2.41	1.50	1.74	8.60	9.98
4	Pb	13.20	2.30	1.57	1.82	9.37	10.87
	Sn(*w*)	14.48	2.23	1.62	1.88	10.03	11.64

[a]The dimensionless radius parameter is defined as $r_s = r_0/a_H$, where a_H is the first Bohr radius and r_0 is the radius of a sphere that contains one electron.

4. ***Fermi gases in astrophysics.*** (a) Given $M_\odot = 2 \times 10^{33}$ g for the mass of the Sun, estimate the number of electrons in the Sun. In a white dwarf star this number of electrons may be ionized and contained in a sphere of radius 2×10^9 cm; find the Fermi energy of the electrons in electron volts. (b) The energy of an electron in the relativistic limit $\epsilon \gg mc^2$ is related to the wavevector as $\epsilon \cong pc = \hbar kc$. Show that the Fermi energy in this limit is $\epsilon_F \approx \hbar c\,(N/V)^{1/3}$, roughly. (c) If the above number of electrons were contained within a pulsar of radius 10 km, show that the Fermi energy would be $\approx 10^8$ eV. This value explains why pulsars are believed to be composed largely of neutrons rather than of protons and electrons, for the energy release in the reaction $n \rightarrow p + e^-$ is only 0.8×10^6 eV, which is not large enough to enable many electrons to form a Fermi sea. The neutron decay proceeds only until the electron concentration builds up enough to create a Fermi level of 0.8×10^6 eV, at which point the neutron, proton, and electron concentrations are in equilibrium.

5. ***Liquid He^3.*** The atom He^3 has spin $\frac{1}{2}$ and is a fermion. The density of liquid He^3 is 0.081 g cm^{-3} near absolute zero. Calculate the Fermi energy ϵ_F and the Fermi temperature T_F.

6. ***Frequency dependence of the electrical conductivity.*** Use the equation $m(dv/dt + v/\tau) = -eE$ for the electron drift velocity v to show that the conductivity at frequency ω is

$$\sigma(\omega) = \sigma(0)\left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2}\right) , \tag{62}$$

where $\sigma(0) = ne^2\tau/m$.