Notes for Lecture 14

Interaction of light and matter

We have explored the time dependent perturbation formalism completely in the last lecture. Here, we shall explore a parallel topic—how light interacts with charge.

14.1 Interaction between charge and light (CM)

Let us consider the good old *classical* mechanics for a little while.

Assume that there is a particle with mass m and charge q, and that this particle interacts with light.

The classical Hamiltonian for this problem is given by

$$H(\vec{r},\vec{p},t) = \frac{(\vec{p}-q\vec{A})^2}{2m} + q\phi \tag{14.1}$$

where the vector potential $\vec{A}(\vec{r},t)$ and the scalar potential $\phi(\vec{r},t)$ give the electromagnetic fields

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \tag{14.2}$$

$$\vec{B} = \nabla\times\vec{A} \tag{14.3}$$

Note that both $\vec{A}$ and ϕ are dependent on $\vec{r}$ and t, in general.

From E&M, let us recall that the above definitions of $\vec{A}$ and ϕ are not unique, since for any differentiable function $f(\vec{r},t)$, we can transform potentials as

$$\vec{A} \to \vec{A} + \nabla f \tag{14.4}$$

$$\phi \to \phi - \frac{\partial f}{\partial t} \tag{14.5}$$

This transformation is called a **gauge transformation**, and choosing a convenient function f is called "fixing/choosing a gauge." Since force fields $\vec{E}$ and $\vec{B}$ are independent of gauge, physics must be independent of gauge[1].

Where does Eq. 14.1 come from? One could "derive" this Hamiltonian, by constructing a Lorentz-invariant Lagrangian first (cf. Landau and Lifshitz, *The Classical Theory of Fields*, pages 45,46; the potential energy due to four potential $(\phi,\vec{A})$ is given by $q\phi - q\vec{A}\cdot\vec{v}$). However, it also suffices to show that the above Hamiltonian does give the correct equation of motion

$$m\dot{\vec{v}} \ = \ q\vec{E} + q\vec{v}\times\vec{B} \tag{14.6}$$

This is left for your work (homework). Please do not misunderstand that the above Hamiltonian means that the potential energy V is simply given by $q\phi$. It is not. The Lagrangian of this problem is given by $L = T - V$ where $V = q\phi - q\vec{A}\cdot\vec{v}$. This is an example where the potential energy V is velocity dependent. Now, from this Lagrangian one can show readily that the canonical momentum is given by $\vec{p} \ = m\vec{v} + q\vec{A}$, and that $H = \dot{\vec{r}}\cdot\vec{p} - L$ becomes the above form.

Other than the formal proof that the Hamiltonian must be written as Eq. 14.1, here we will carry out a simple thought experiment ("Gedankenexperiment") to illustrate a crucial point, i.e., the *canonical momentum* conjugate to $\vec{r}$ in the presence of the electromagnetic field is given, indeed, by

$$\vec{p} \ = \ m\vec{v} + q\vec{A} \tag{14.7}$$

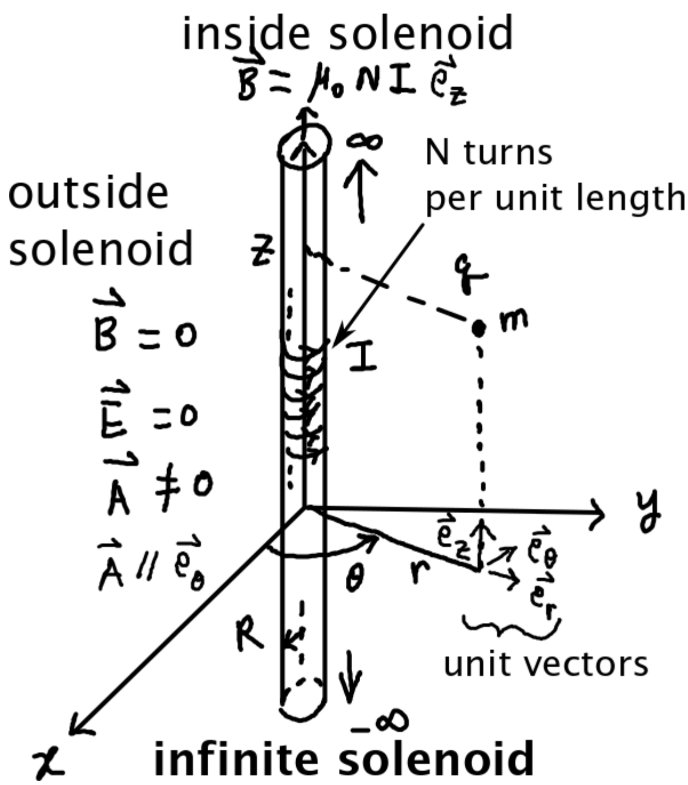
This example is quite important, as we will see later in this course.

Let us suppose that there is an infinitely long solenoid of radius R. We take the axis of the solenoid to be the z axis. We shall use the *cylindrical coordinate system* (r,θ,z), with unit vectors defined as $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$, respectively. Let us assume that the solenoid carries a constant current I and has N turns per unit length. Then, by Ampere's law, we have $\vec{B} = \mu_0 NI\,\vec{e}_z$ (where $\vec{e}_z$ is the unit vector along the z axis) inside the solenoid, and zero outside the solenoid. Let us consider a circle in the xy

[1] This is also largely true in quantum mechanics. Thus, "gauge invariance" or "gauge symmetry" of physical laws is an important symmetry. However, in a superconducting state, the gauge symmetry is broken, while the Hamiltonian is gauge-invariant. Such a thing is called a "(spontaneous) broken symmetry." The Higgs mechanism also relies on such a broken gauge symmetry.

plane, centered at the origin and with radius r. The magnetic field enclosed by this circle, $\Phi(r)$, is readily given as

$$\Phi(r) = \begin{cases} \pi\mu_0 N I r^2 & \text{if } r < R \\ \pi\mu_0 N I R^2 & \text{if } r > R \end{cases} \tag{14.8}$$



Let us ask a question: what is the vector potential $\vec{A}$? The same magnetic flux can be expressed in terms of $\vec{A}$ as, using the Stoke's theorem,

$$\Phi(r) = \int_S d\vec{\sigma} \cdot \vec{B} = \int_S d\vec{\sigma} \cdot \nabla \times \vec{A} = \oint_C d\vec{l} \cdot \vec{A} = 2\pi r A_\theta(r) \tag{14.9}$$

where in the last step the polar component in the xy plane, A_θ, is taken to be independent of θ, consistent with the cylindrical symmetry of the problem. Therefore, we can[2] take

$$\vec{A} = A_\theta(r)\,\vec{e}_\theta = \begin{cases} \mu_0 N I \cdot \dfrac{r}{2}\,\vec{e}_\theta & \text{if } r < R \\ \mu_0 N I \cdot \dfrac{R^2}{2r}\,\vec{e}_\theta & \text{if } r > R \end{cases} \tag{14.10}$$

We will use what we have figured out so far in another important problem later. The really crucial thing here is that the vector potential $\vec{A}$ can *never be zero* anywhere in space (except at one point, the origin). However, we *can* take ϕ to be zero for this problem (see footnote 2).

Now, we consider the motion of our particle with mass m and charge q *outside* the solenoid. Notice that there is absolutely no field, $\vec{B}$ or $\vec{E}$, outside the solenoid. Thus, our particle is free, and experiences no force at all.

Let our particle just moves in the empty space. Let us also assume that the particle is removed sufficiently far from the solenoid and so it "does not see/know the solenoid." If you like, you can imagine that the particle in question is a small bug (which is OK as a particle since we are doing classical mechanics) that somehow acquired some net charge q. Also, let us assume that this bug does not see beyond a few inches, and so it could not care less about the solenoid far away from it.

If this is all we have, then what will happen? Basically, nothing. The bug will move at a constant velocity since there is no force whatsoever. It is a simple Newton's second law problem. However, we feel a bit uneasy about this situation. Why is the Hamiltonian so elaborate in this case? Eq. 14.1 with $\vec{A}$ given by Eq. 14.10 (and $\phi = 0$) seems to be such an overly complicated Hamiltonian for a constant velocity motion!

Indeed, one can say the following for this situation. From the velocity point of view, the problem is extremely simple. However, from the momentum point of view, this problem seems, and is, very complicated! First of all, the momentum $\vec{p}$ is *not* conserved, since (by Hamilton's equation) $\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}$, and since H *depends* on $\vec{r}$ through $A(\vec{r})$. **Thus, we have an extraordinary situation where the velocity is constant, but the momentum is not!** Of course, this is obvious just by glancing at the two equations, Eqs. 14.6 and 14.7, while keeping in mind that $\vec{B} = \vec{E} = 0$ and $\vec{A} \neq 0$.

You might wonder whether there is any measurable consequence of the vector potential that appears in the momentum for this very setup. In classical mechanics,

[2]Note that this vector potential is not unique, since the electromagnetic potentials are arbitrary up to a gauge transformation. Here, we have taken the "Coulomb gauge" $\nabla \cdot \vec{A} = 0$, which is a convenient gauge to use when the charge density source for the electromagnetic field is time-independent. In the current problem, the charge density source is, in fact, zero. A consequence of taking this Coulomb gauge for the current problem is that $\phi = 0$.

the answer is no. In quantum mechanics, the answer is yes, as we shall see later in the course[3].

However, not surprisingly, the measurable quantities are derived[4] from the "differences"[5] of the vector potential, and are eventually connected to the force fields. Let us, in fact, consider a situation which involves a time-dependent change of $\vec{A}$, for which even the classical problem that we are considering at the moment has a measurable consequence.

So now, imagine that, at one moment, the current through the solenoid changes by ΔI during a small time interval Δt. We can imagine that a small change like this can be caused by a variety of things. For instance, perhaps there is a small temperature drop and thus the resistance decreased, causing I to go up a bit for a given fixed voltage supplied. Does anything happen to the particle? The answer is yes! Due to the change in current, we get a small change in the vector potential $\Delta A_\theta(r) \propto \Delta I$ during time Δt. This means that there *is* an electric field temporarily due to the Faraday effect: $\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\Delta \vec{A}}{\Delta t}$ during Δt. However, notice that $\nabla \times \vec{A} = 0$ all the time, and so there is no $\vec{B}$ field. By Newton's law (Eq. 14.6, which says $m\dot{\vec{v}} = \vec{F}$, *not* $\dot{\vec{p}} = \vec{F}$), then, we get

$$m\Delta\vec{v} \; = \; q\vec{E}\Delta t \; = \; -q\Delta\vec{A} \tag{14.11}$$

What is extraordinary about this equation is that it can be rewritten as

$$\Delta\vec{p} \equiv \Delta\left(m\vec{v} + q\vec{A}\right) \; = \; 0 \tag{14.12}$$

So, now we have an extraordinary situation where the velocity changes, while the momentum is conserved! Why is the canonical momentum conserved during Δt? Effectively, the position dependence of the Hamiltonian, Eq. 14.1, is frozen in, since we have assumed that during Δt the particle does *not* move[6]. Therefore, $\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = 0$, during Δt. And, of course, this type of conservation law is something by which the canonical momentum is given a deep meaning.

[3] Some of you might shout, at this point, "the momentum is not gauge invariant!" This is quite true. What we will see later is that the measurable consequence in quantum mechanics is gauge-invariant. Also, note that the formalism of quantum mechanics *allows* for the introduction of an arbitrary gauge, which as we will see later corresponds to an additive phase (not a constant but a function) in the wave function—read Dirac, *The Principles of Quantum Mechanics*, pages 92,93.

[4] This is assuming that we do not have some extraordinary broken gauge symmetry state. Read footnote 1.

[5] These may very well be integrals of some sort, not just the difference between two points. So, the quotation marks.

[6] More precisely, we are assuming that the change of $\vec{A}$ due to any small motion is much smaller than $\Delta\vec{A}$ due to the current change.

Therefore, when a charged particle is moving in a vector potential, $\vec{A}$, its momentum consists of two parts. The ordinary part $m\vec{v}$, and the **field momentum** $q\vec{A}$.

One must consider the field momentum as the way that the charged particle "acknowledges/knows" the presence of the vector potential, just as the scalar potential energy term $q\phi$ is the way that the charged particle "acknowledges/knows" the presence of the scalar potential. This does not mean that measurable quantities are dependent on absolute (as opposed to relative, not as opposed to signed) values of $\vec{A}$ and ϕ; on the contrary, measurable quantities depend only on "differences" (see footnote 5) of these quantities, which eventually get connected to force fields.

14.2 Interaction between charge and light (QM)

Having established how light interacts with charge in classical mechanics, all we need to do for quantum mechanics is to quantize operators.

$$\hat{H} = \frac{(\hat{\vec{p}} - q\hat{\vec{A}})^2}{2m} + q\hat{\phi} \qquad \text{for a } \textit{safer} \text{ form, see Eq. 14.14 below} \tag{14.13}$$

There is one thing to be careful about. Since $\hat{\vec{p}}$ and $\hat{\vec{A}}$ generally do not commute, when expanding this equation, we must rewrite it as

$$\hat{H} = \frac{\hat{p}^2}{2m} - \frac{q}{2m}\left(\hat{\vec{p}} \cdot \hat{\vec{A}} + \hat{\vec{A}} \cdot \hat{\vec{p}}\right) + \frac{q^2}{2m}\hat{A}^2 + q\hat{\phi} \tag{14.14}$$

$$\hat{\vec{p}} \doteq -i\hbar\nabla \tag{14.15}$$

However, if we use the Coulomb gauge (i.e. the transverse gauge), $\nabla \cdot \vec{A} = 0$, which is always a possible choice and is particularly useful for the case when the charge density source for the electromagnetic wave is static, then we get

$$\hat{\vec{p}} \cdot \hat{\vec{A}} \doteq -i\hbar\nabla \cdot \vec{A} = -i\hbar\left(\nabla \cdot \vec{A}\right) - i\hbar\vec{A} \cdot \nabla = -i\hbar\vec{A} \cdot \nabla \;\overset{\cdot}{=}\; \hat{\vec{A}} \cdot \hat{\vec{p}}$$

where in the second expression, $\nabla \cdot \vec{A}$ is an operator for which ∇ will operate on $\vec{A}$ *or* (by the product rule of differential calculus) on any function that may come on the right side of $\vec{A}$, while in the third expression, $\left(\nabla \cdot \vec{A}\right)$ is simply a function, not a differential operator any more for any function that may appear on the right side. So we get

$$\hat{H} = \frac{\hat{p}^2}{2m} - \frac{q}{m}\hat{\vec{A}} \cdot \hat{\vec{p}} + \frac{q^2}{2m}\hat{A}^2 + q\hat{\phi} \qquad \text{Coulomb gauge } \nabla \cdot \vec{A} = 0 \tag{14.16}$$

We will use this equation exclusively to discuss the interaction of charge with light.

14.3 Interaction between electron and light

For an electron, we use $q = -e$ and $m = m_e$, and we get

$$\hat{H} = \frac{\hat{p}^2}{2m_e} + \frac{e}{m_e}\hat{\vec{A}}\cdot\hat{\vec{p}} + \frac{e^2}{2m_e}\hat{A}^2 - e\hat{\phi} \qquad \text{Coulomb gauge } \nabla\cdot\vec{A} = 0 \tag{14.17}$$

For light propagating in vacuum coming and hitting an electron bound to an atom, ϕ would correspond to the Coulomb potential for the atomic binding, and $\vec{A}$ would correspond completely to the time dependent vector potential for the light. The scalar potential contribution from the light, ϕ_l, can be taken to be zero within the Coulomb gauge, where ϕ_l satisfies the Poisson equation $-\nabla^2\phi_l = \frac{\rho_s}{\epsilon_0}$, where ρ_s is the source charge density for the light wave: we assume that any charge density that created the light wave is very far away[7] so that ϕ_l vanishes at the atom.

Then, the above equation can be rewritten as

$$\hat{H}_0 = \frac{\hat{p}^2}{2m_e} - e\hat{\phi} \tag{14.18}$$

$$\hat{H}_1 = \frac{e}{m_e}\hat{\vec{A}}\cdot\hat{\vec{p}} + \frac{e^2}{2m_e}\hat{A}^2 \tag{14.19}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \tag{14.20}$$

Here, $\hat{H}_0$ is the "atomic potential" for which we assume that we know the answer. $\hat{H}_1$ is the time dependent perturbation term.

From Maxwell's equation, within the Coulomb gauge and with no current source nearby, the equation for $\vec{A}$ comes out as (derivation left for your exercise)

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \tag{14.21}$$

which is a common wave equation. Assuming a traveling wave form

$$\vec{A} \propto \exp(\pm i(\vec{k}\cdot\vec{r} - \omega t))$$

and plugging it into the wave equation, we get the dispersion relation

$$\omega = ck \qquad k \equiv |\vec{k}|, \quad \textbf{dispersion relation} \tag{14.22}$$

and the Coulomb gauge condition $\nabla\cdot\vec{A} = 0$ becomes

$$\vec{k}\cdot\vec{A} = 0 \tag{14.23}$$

[7]This would be valid, e.g., if the distance between the light source and the atom is macroscopic, like a meter, and there is no spurious net charge for any equipment (i.e. they are well-grounded).

which is why the Coulomb gauge is called the transverse gauge.

Then, the general solution for the light wave with wave vector $\vec{k}$ and frequency ω is given by

$$\vec{A} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{A}_0^* e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{14.24}$$

where the requirement that $\vec{A}$ is real has been imposed. The Coulomb gauge condition becomes

$$\vec{k}\cdot\vec{A}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{k}\cdot\vec{A}_0^*\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)} = 0$$

As the two functions e^{ix} and e^{-ix} are *independent* functions of $x = \vec{k}\cdot\vec{r} - \omega t$, we see that both $\vec{k}\cdot\vec{A}_0$ and $\vec{k}\cdot\vec{A}_0^*$ must vanish. It is sufficient to require that

$$\vec{k}\cdot\vec{A}_0 = 0 \qquad \text{Coulomb gauge} \tag{14.25}$$

So, for a given $\vec{k}$, we get two independent directions for $\vec{A}_0$. Define a unit vector, $\vec{\varepsilon}$, such that[8]

$$\vec{\varepsilon}\cdot\vec{\varepsilon} = 1 \tag{14.26}$$

$$\vec{\varepsilon}\cdot\vec{k} = 0 \tag{14.27}$$

We shall also assume that $\vec{A}_0 = A_0\vec{\varepsilon}$, with A_0 being a positive number. This corresponds to choosing a fixed initial phase for the light field. While this may seem a bit arbitrary, it does not affect our final results. Then, the vector potential is given by

$$\vec{A} = \vec{\varepsilon}\, A_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{\varepsilon}\, A_0\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{14.28}$$

Now, applying Eqs. 14.2 and 14.3, we get

$$\vec{E} = i\omega\,\vec{\varepsilon}\, A_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + c.c. \qquad c.c. = \text{complex conjugate} \tag{14.29}$$

$$\vec{B} = i\vec{k}\times\vec{\varepsilon}\, A_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + c.c. \tag{14.30}$$

The energy density of this light field is given by (from E&M)

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} \tag{14.31}$$

[8]It is also possible to define $\vec{\varepsilon}$ as a complex unit vector, to deal with circularly polarized light. This requires a trivial generalization, which I leave to readers.

Let us call $\bar{u}$ the average of u over a period. For $\bar{u}$, any oscillatory term vanishes, when we calculate the time average of E^2 or B^2.

$$\bar{u} = A_0^2 \left(\epsilon_0 \omega^2 + \frac{k^2}{\mu_0} \right) \tag{14.32}$$

$$= 2\epsilon_0 A_0^2 \omega^2 \qquad \because \quad \mu_0 \epsilon_0 = c^{-2}, \quad \omega = ck \tag{14.33}$$

Now, suppose that there are N photons (of particular polarization, wave vector and frequency) in a volume $\mathcal{V}$. Then, this energy density must be equal to $N\hbar\omega/\mathcal{V}$. Necessarily, we are assuming that $N \gg 1$, since we are treating the E&M field within classical mechanics, using Maxwell equations (see Section 14.4 below). Therefore, we get

$$A_0 = \sqrt{\frac{N\hbar}{2\epsilon_0 \omega \mathcal{V}}} \tag{14.34}$$

Note that A_0 determines the typical magnitude of $\vec{A}$. In the low density limit, and in the first approximation for coupling constant e for the interaction between the electron and the light, we see that the first term in Eq. 14.19 will dominate over the second term. So, we ignore the second term, and collect all results that we gathered so far

$$\hat{H}_1 = \hat{V} \exp(-i\omega t) + \hat{V}^\dagger \exp(i\omega t) \tag{14.35}$$

where

$$\hat{V} = \frac{eA_0}{m_e} e^{i\vec{k}\cdot\hat{\vec{r}}} \vec{\varepsilon} \cdot \hat{\vec{p}} = \frac{e}{m_e} \sqrt{\frac{N\hbar}{2\epsilon_0 \omega \mathcal{V}}} e^{i\vec{k}\cdot\hat{\vec{r}}} \vec{\varepsilon} \cdot \hat{\vec{p}} \qquad \text{classical light} \tag{14.36}$$

14.4 Interaction between electron and photon

This section is somewhat advanced, while it should be read by everyone.

So far, we have somewhat avoided using the word "photon," and instead have used the word "light" for the most part. This is quite the right thing to do, since, if you look carefully at what we have done so far, we did not do any quantum mechanics on light! By light, we have simply meant a classical electromagnetic wave so far.

That is, we have been doing "classical optics" so far. On the other hand, the motion of the electron (or a charged particle that interacts with light) *was* quantized. The only place where we used the "quantum optics" or "quantum electrodynamic (QED)" concept of photon is when we asserted just before Eq. 14.34 that the total energy of the light is given by $N\hbar\omega$. However, still, this was done in the classical regime, as we have not introduced any *quantum mechanical operator* for treating the light/photon degrees of freedom.

Here in this section, we will see how the quantum nature of light—the photon concept—can be introduced with minimal effort, by making two assumptions. (1) A photon mode is a simple harmonic oscillation mode. (2) The $\sqrt{N}$ factor occurring in A_0 changes to the ladder up or down operator depending on whether it corresponds to the photon emission or absorption term. These assumptions follow the standard QED formalism—however, notice that what we have learned so far certainly make these assumptions seem very plausible and very well motivated. These two assumptions are fundamental postulates of quantum optics.

So, in the quantum mechanical treatment of light, the only change that occurs to the formulae that we have developed so far is that A_0 now becomes an operator

$$\hat{A}_0 = \sqrt{\frac{\hbar}{2\epsilon_0\omega\mathcal{V}}}\,\hat{a}_{\vec{k},\vec{\varepsilon}}, \quad \hat{A}_0^\dagger = \sqrt{\frac{\hbar}{2\epsilon_0\omega\mathcal{V}}}\,\hat{a}^\dagger_{\vec{k},\vec{\varepsilon}} \qquad \text{photon—quantized light} \tag{14.37}$$

where $\hat{a}_{\vec{k},\vec{\varepsilon}}$ and $\hat{a}^\dagger_{\vec{k},\vec{\varepsilon}}$ are ladder operators for the simple harmonic motion that corresponds to the light wave at $\vec{k}$ and $\vec{\varepsilon}$. Thus, Eq. 14.24 is replaced by

$$\hat{\vec{A}} = \vec{\varepsilon}\hat{A}_0 e^{i(\vec{k}\cdot\hat{\vec{r}}-\omega t)} + \vec{\varepsilon}\hat{A}_0^\dagger e^{-i(\vec{k}\cdot\hat{\vec{r}}-\omega t)} \qquad \text{photon—quantized light} \tag{14.38}$$

Then,

$$\hat{V} = \frac{eA_0}{m_e}\,e^{i\vec{k}\cdot\hat{\vec{r}}}\,\vec{\varepsilon}\cdot\hat{\vec{p}} = \frac{e}{m_e}\sqrt{\frac{\hbar}{2\epsilon_0\omega\mathcal{V}}}\,e^{i\vec{k}\cdot\hat{\vec{r}}}\,\vec{\varepsilon}\cdot\hat{\vec{p}}\,\hat{a}_{\vec{k},\vec{\varepsilon}} \qquad \text{photon—quantized light} \tag{14.39}$$

and $\hat{V}^\dagger$ is its Hermitian conjugate. Note that a photon ladder operator and an electron operator ($\hat{\vec{r}}$ or $\hat{\vec{p}}$) are completely commuting, since they live in completely different Hilbert spaces. When the simple harmonic oscillator ladder operators are used to define the photon field like here, the lowering operator is called **the destruction operator** and the raising operator is called **the creation operator**. From our knowledge of the simple harmonic oscillator, we get

$$\hat{V}\left|N_{\vec{k},\vec{\varepsilon}}\right\rangle = \frac{e}{m_e}\sqrt{\frac{N\hbar}{2\epsilon_0\omega\mathcal{V}}}\,e^{i\vec{k}\cdot\hat{\vec{r}}}\,\vec{\varepsilon}\cdot\hat{\vec{p}}\left|(N-1)_{\vec{k},\vec{\varepsilon}}\right\rangle \tag{14.40}$$

$$\hat{V}^\dagger\left|N_{\vec{k},\vec{\varepsilon}}\right\rangle = \frac{e}{m_e}\sqrt{\frac{(N+1)\hbar}{2\epsilon_0\omega\mathcal{V}}}\,e^{i\vec{k}\cdot\hat{\vec{r}}}\,\vec{\varepsilon}\cdot\hat{\vec{p}}\left|(N+1)_{\vec{k},\vec{\varepsilon}}\right\rangle \tag{14.41}$$

where $\left|N_{\vec{k},\vec{\varepsilon}}\right\rangle$ is a state of N photons with wave vector $\vec{k}$ and polarization $\vec{\varepsilon}$. Namely, N (= 0, 1, 2, ...) corresponds to the familiar quantum number n (= 0, 1, 2, ...)

of a simple harmonic oscillator. Note that, as we discussed in LN 13, the $\hat{V}$ term corresponds to the absorption, since when $\hat{V}$ is acting on an N photon state, we get an $N-1$ photon state (it *destroys* a photon, by the action of $a_{\vec{k},\vec{\varepsilon}}$). Likewise, the $\hat{V}^\dagger$ term corresponds to the emission, since the photon number increases by 1, by its action (it *creates* a photon, by the action of $a^\dagger_{\vec{k},\vec{\varepsilon}}$). What is a notable difference between these quantum mechanical (i.e. quantum electro-dynamic = QED) results and the classical E&M result (Eq. 14.36)? It is the $\sqrt{N+1}$ factor in the QED result for the photon creation part, as opposed to the $\sqrt{N}$ factor in the corresponding classical E&M result (Eq. 14.36).

What does this difference actually mean? It means that in the classical picture, **only stimulated emission** is possible, since if $N = 0$ in the initial state, $\hat{V}^\dagger$ simply vanishes. Not so, in QED!! By the simple harmonic nature of the photon, we get that **the spontaneous emission** is possible! This is because $V^\dagger \left| N_{\vec{k},\vec{\varepsilon}} \right\rangle$ does not vanish any more for $N = 0$, thanks to the $\sqrt{N+1}$ factor. You can consider the actual existence of the spontaneous emission in nature as proof that photons have zero point motions, since otherwise how can no light induce an emission?

Note, however, that the QED result (for emission; absorption result is already identical with the classical result) converges to the classical result when the quantum number N is high, consistent with the correspondence principle.

Lastly, you might wonder—how did physicists actually figure out that the photon is described by a simple harmonic oscillator, and how can we know more about the oscillator? This question is good, but the answer is not clear. The above travelling wave solution does suggest that a simple harmonic oscillator might be an underlying cause of a photon. And, indeed, when we *postulate* that a photon state *is* a simple harmonic oscillator state, where the eigenvalue of the number operator, $\hat{a}^\dagger_{\vec{k},\vec{\varepsilon}}\hat{a}_{\vec{k},\vec{\varepsilon}}$, is interpreted as the number of photons, then everything comes out wonderfully, explaining everything that we see in experiments. However, if one asks "what is vibrating to make photons?", we have no verifiable/falsifiable answer yet. You can compare this situation with that of a sound wave. When a sound wave is quantized, we call it phonon. Its mathematical structure is exactly like that for the photon—it is a simple harmonic oscillator per mode (wave vector and polarization)! But, we *do* know that sound wave is the state of vibrating atoms, while atoms themselves are not in any sense to be called a sound wave.