- State $\quad |\alpha\rangle = \langle\alpha|^\dagger \quad \langle\alpha| = |\alpha\rangle^\dagger$

- Operator
  - Hermitian $\quad \hat{h} = \hat{h}^\dagger$ → Observable
  - Unitary $\quad \hat{U}^{-1} = \hat{U}^\dagger$ → Symmetry op. / Coordinate transformation in the Hilbert space / Basis changing op.

Note

$$(|\alpha\rangle\langle\beta|)^\dagger = |\beta\rangle\langle\alpha|$$

$$(\hat{O}_1 \hat{O}_2)^\dagger = \hat{O}_2^\dagger \hat{O}_1^\dagger$$

$$\langle\alpha|\beta\rangle^\dagger = \langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$$

$$\left(c_1|\alpha\rangle + c_2|\beta\rangle\right)^\dagger = \langle\alpha| c_1^* + \langle\beta| c_2^* = c_1^*\langle\alpha| + c_2^*\langle\beta|$$

$$(\hat{O}|\alpha\rangle)^\dagger = \langle\alpha|\hat{O}^\dagger$$

$$\langle\alpha|\hat{O}|\beta\rangle^\dagger = \langle\beta|\hat{O}^\dagger|\alpha\rangle = \langle\alpha|\hat{O}|\beta\rangle^*$$

• Resolution of identity

<u>Any</u> hermitian operator $\hat{h}$

its e-vectors $|h\rangle$

$$\sum_h |h\rangle\langle h| = 1$$

if h discrete

$$\int dh \, |h\rangle\langle h| = 1$$

if h continuous

$$\langle h | h' \rangle = \delta_{h h'}$$

$$\langle h | h' \rangle = \delta(h - h')$$

• Measurement principle

An observable $\hat{h}$

What do we get when we measure $\hat{h}$ on $|\Psi\rangle$?

<u>Ans.</u> One of h ← e-value

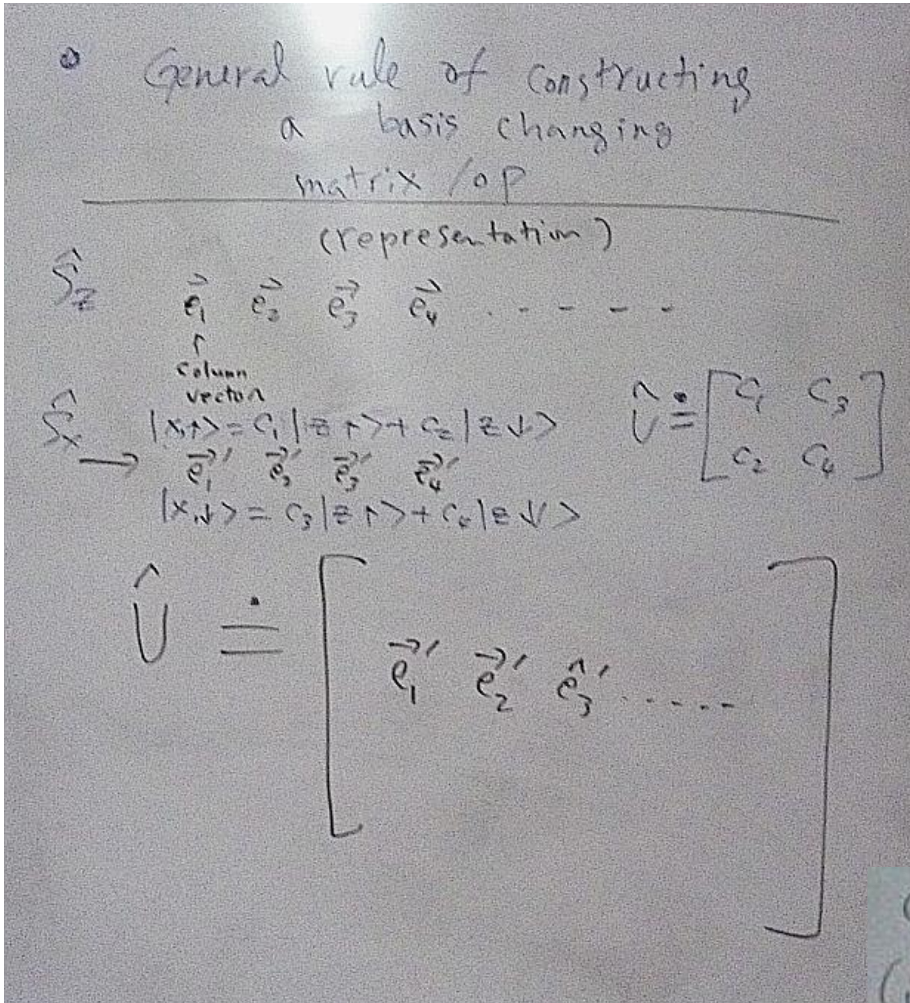
$$(\hat{h} |h\rangle = h |\hat{h}\rangle)$$

Probability?

$$|\langle h | \Psi \rangle|^2$$ ← discrete

or $$|\langle h | \Psi \rangle|^2 \, dh$$ ← continuum

$[h, h+dh]$

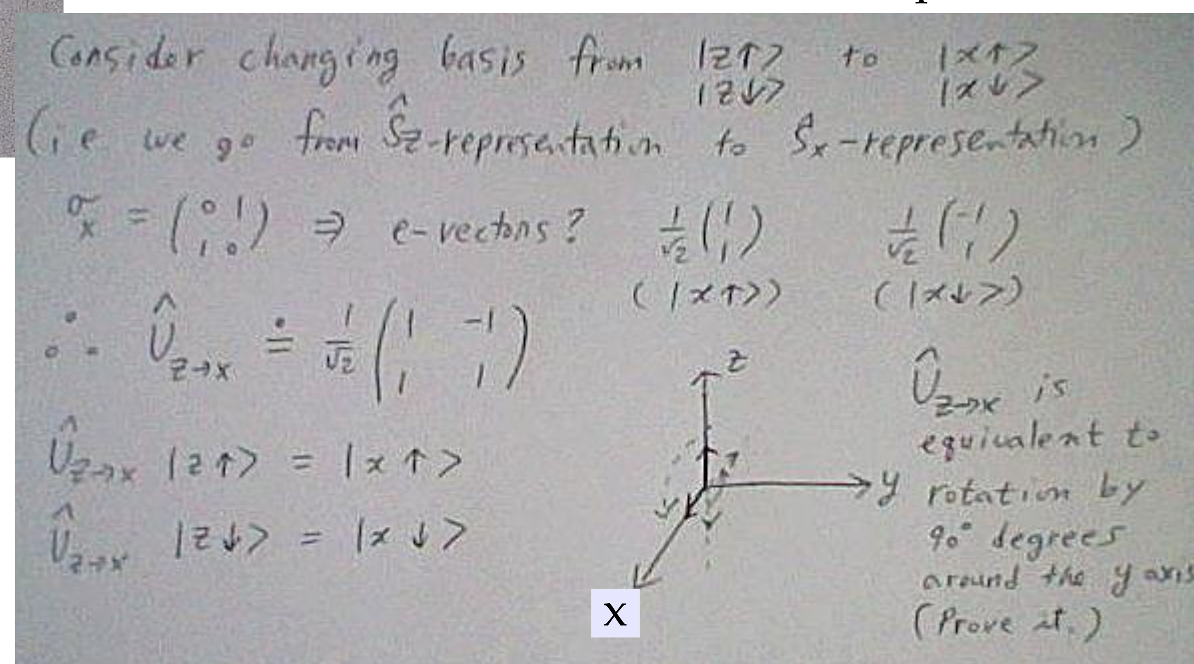


This was a general comment on how to construct a basis changing matrix, which includes a unitary matrix.

This knowledge is useful in all contexts.

The point is that if you knew how each basis vector transforms, then finding the transformation matrix is as easy as simply juxtaposing the transformed basis vectors (taken to be *column* vectors, importantly).

Below is an additional note on this point.



A unitary transformation is a basis changing transformation from an orthonormal basis to another orthonormal basis.

- Representation

Take a natural basis

$\langle m|n\rangle = \delta_{m,n}$ if discr.

$\langle m|n\rangle = \delta(m-n)$ if cont.

$$\Rightarrow \quad |\Psi\rangle \doteq \langle m|\Psi\rangle$$

$$\hat{O} \doteq \langle n|\hat{O}|n\rangle$$

$$\hat{O} \doteq \begin{array}{c|c} & |0\rangle \ |1\rangle \ |2\rangle \ \cdots \\ \begin{array}{c}\langle 0| \\ \langle 1| \\ \langle 2| \\ \vdots\end{array} & \begin{bmatrix} \langle 0|\hat{O}|0\rangle & \cdots \\ \langle 1|\hat{O}|0\rangle & \ddots \\ \vdots & \end{bmatrix} \end{array}$$

cr.

ont.

- Symmetry ops

	infinitesimal	Finite
Time evolution	$\hookrightarrow = 1 - i \frac{\hat{H}\, dt}{\hbar}$	$e^{-i\hat{H}t/\hbar}$ (if $\frac{\partial \hat{H}}{\partial t} = 0$)
Rotation	$\hookrightarrow 1 - i \frac{\hat{L}_\theta\, d\theta}{\hbar}$	$e^{-i\hat{L}_\theta \theta/\hbar}$
Translation	$\hookrightarrow 1 - i \frac{\hat{p}\, dx}{\hbar}$	$e^{-i\hat{p}x/\hbar}$

- Conservation

$\frac{\partial \hat{H}}{\partial t} = 0 \Leftrightarrow \hat{H}$ conserved (energy cons.)

$[\hat{L}_\theta, \hat{H}] = 0 \Leftrightarrow \hat{H}$ rot invariant $\Leftrightarrow \hat{L}_\theta$ conserved

$[\hat{p}, \hat{H}] = 0 \Leftrightarrow \hat{H}$ trans. invariant $\Leftrightarrow \hat{p}$ conserved

$[\hat{A}, \hat{B}] = 0 \Rightarrow$ compatible operators

$\hat{A}, \hat{B}$ are compatible.

- Conjugate variables

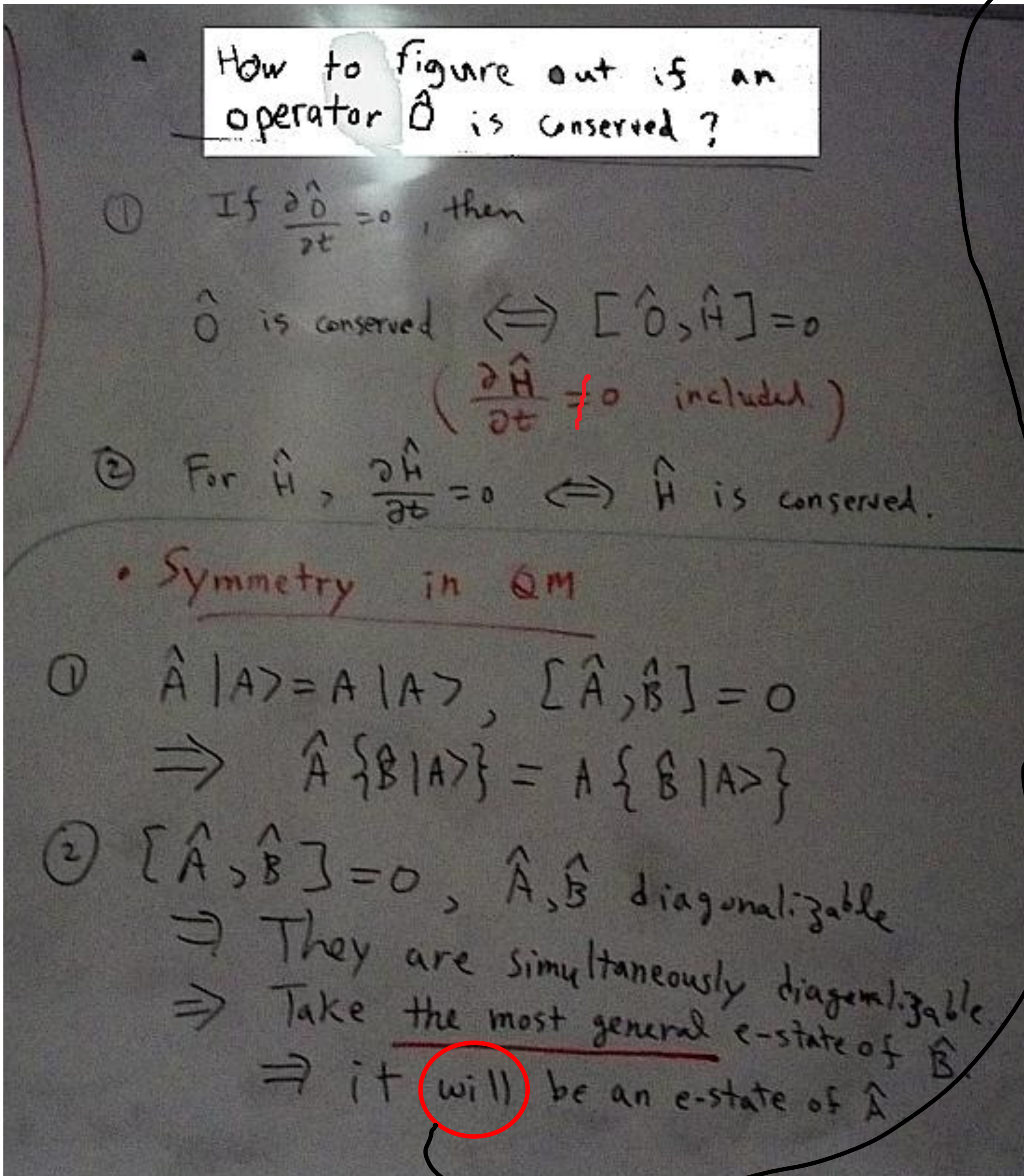
$\hat{L}_\theta \doteq -i\hbar \frac{\partial}{\partial \theta}$

$\hat{p}_x \doteq -i\hbar \frac{\partial}{\partial x}$

$\hat{H}$ "$\doteq$" $i\hbar \frac{\partial}{\partial t}$ (Schrödinger eq.)

- Conservation?

$\hat{O}$ is conserved if $\frac{d\langle \hat{O} \rangle}{dt} = 0$ for any state $|\Psi\rangle$



It "will" be an e-state, but more work is necessary. (So, "can" may be better.)

The point is that if there is a degeneracy for $\hat{B}$, then choosing only one of the degenerate states "will most likely not" given an eigenstate of $\hat{A}$, while choosing the most general form "will" lead to an eigenstate of $\hat{A}$ with some additional work. (The additional work left to do is called "block diagonalization." Sections 7.1 and 7.2)

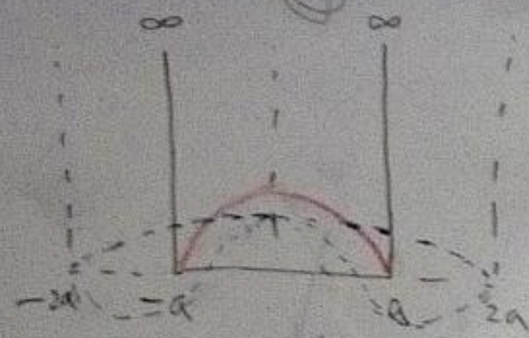
Example.

Free particle in 1D. Take $\hat{A}$=free particle Hamiltonian. Parity symmetry ($\hat{B}$=parity).

Choosing a random even state (or an odd state) will not likely to give an energy e-state: e.g. x^2 is not an energy eigenstate. However, taking the energy eigenstate as a general evenstate and then solving the eigenvalue equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$$

gives $\psi = \mathrm{A}\cos(kx)$.



$|\Psi\rangle$

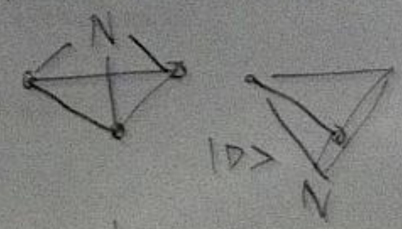
What do we get if we measure E ?

In principle, any E_n for the new well !

$$P_n = \left|\underbrace{\langle E_n|\Psi\rangle}\right|^2$$

$|\langle E_n|\Psi\rangle|$ max for the ground state

$|U\rangle$



$|D\rangle$

degenerate

– accidental degeneracy?

– yes.

$$|e\rangle = \frac{1}{\sqrt{2}}\left(|U\rangle + |D\rangle\right) \leftarrow \text{parity } +$$

$$|o\rangle = \frac{1}{\sqrt{2}}\left(|U\rangle - |D\rangle\right) \leftarrow \text{parity } -$$

$\hat{P}\,|e\rangle = |e\rangle$

$\{|e\rangle\}$ is closed under $\hat{P}$

$\hat{P}\,|o\rangle = -|o\rangle$

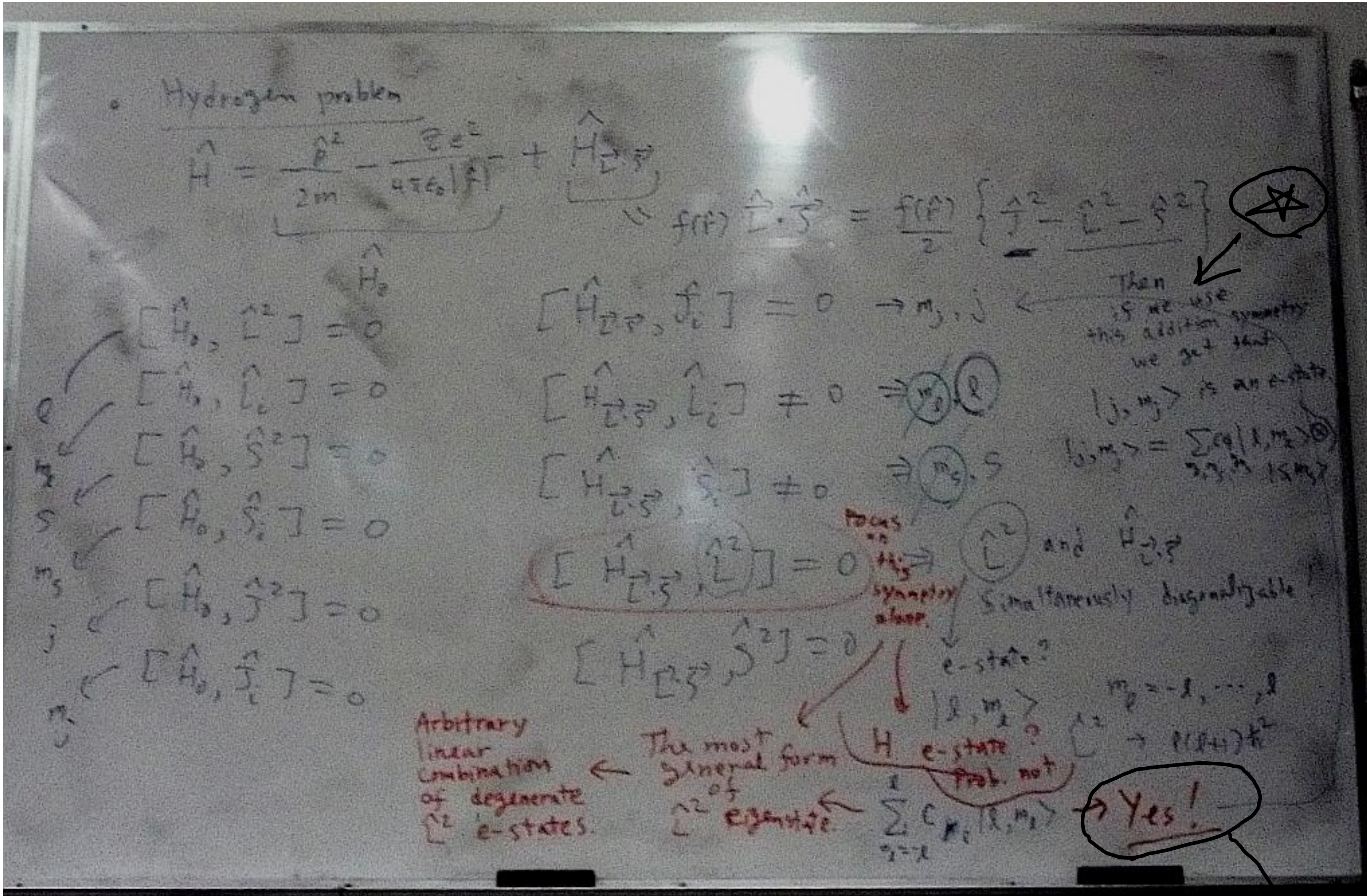
$\uparrow$

parity

$\{|o\rangle\}$ is too!

Symmetry guaranteed degeneracy

$\Rightarrow$ 1!



Note ★ as well as comments in page 6.

- Spin
- SHO

deg. pert. theory

zeroth order e-states $|0^{(0)}\rangle, |1^{(0)}\rangle$
(degenerate)

$E_0 \to \lambda \;\cdots$

$\lambda^* \; E_1 \to$

$\rightarrow$ Diagonalize $\hat{H}$ in the subspace spanned by them

$$\underset{\text{subspace}}{\hat{H}} \doteq \begin{bmatrix} \langle 0^{(0)}|\hat{H}|0^{(0)}\rangle & \langle 0^{(0)}|\hat{H}|1^{(0)}\rangle \\ \langle 1^{(0)}|\hat{H}|0^{(0)}\rangle & \langle 1^{(0)}|\hat{H}|1^{(0)}\rangle \end{bmatrix}$$

$\rightarrow$ e-values : correct up to 1st order

e-vector : zeroth order e-vectors

"good"

$$\begin{bmatrix} 0 & \Delta \\ \Delta & 0 \end{bmatrix} = \begin{matrix} +\Delta & \frac{1}{\sqrt{2}}\left(|0^{(0)}\rangle + |1^{(0)}\rangle\right) \\ -\Delta & \frac{1}{\sqrt{2}}\left(- \right) \end{matrix}$$

if $\Delta < 0$

Additional comments (copied from email message):

(1) You must know Eqs. 3.16, 3.17, 3.18 by heart for this exam. Likewise, Eqs. 4.38 and 4.39 are assumed to be known by heart by you for this exam. They will not be given in my "formula sheet." (This is because it is my goal that you become very familiar with these formulae.)

(2) Please note that in LN 4, page 10, the first box says "|n^(0)\prime>" is the answer. This is important -- when asked to find zeroth order state in a perturbation problem, you are being asked to find "good" zeroth order state (|n^(0)\prime> not |n^(0)> in the notation of LN 4) that diagonalizes the Hamiltonian in the degenerate sub-space. The process of finding these good zero order states is what I called "rebooting the problem" -- it is the process of the diagonalization of the Hamiltonian in the degenerate sub-space.

(3) You must know how to do spin 1/2 problems and simple harmonic oscillator problems in this exam (as we did in some homework problems). For these problems, all basic formulae will be given (Pauli matrices, ladder operators, etc.).

(4) Note that whether the degenerate perturbation theory is applicable or not is a state-specific quation, not a Hamiltonian specific question. The perturbation theory itself is a very state specific theory. We start with a specific state |n^(0)> and ask the question -- what happens to it (see Eqs. 3.6 and 3.13)? Depending on state, one must use a degenerate perturbation theory or a non-degnerate perturbation theory (see HW 2.5 for a very simple example of this).