- State $|\alpha\rangle = \langle\alpha|^{\dagger}$ $\quad \langle\alpha| = |\alpha\rangle^{\dagger}$

- Operator
  - Hermitian $\hat{h} = \hat{h}^{\dagger}$
  - Unitary $\hat{U}^{\dagger} = \hat{U}^{-1}$

- $\left(|\alpha\rangle\langle\beta|\right)^{\dagger} = |\beta\rangle\langle\alpha|$

$$\left(\hat{O}_1 \hat{O}_2 \hat{O}_3\right)^{\dagger} = \hat{O}_3^{\dagger} \hat{O}_2^{\dagger} \hat{O}_1^{\dagger}$$

$$\langle\alpha|\beta\rangle^{\dagger} = \langle\alpha|\beta\rangle^{*} = \langle\beta|\alpha\rangle$$

$$\left(c_1|\alpha\rangle + c_2|\beta\rangle\right)^{\dagger} = \langle\alpha|c_1^{*} + \langle\beta|c_2^{*} = c_1^{*}\langle\alpha| + c_2^{*}\langle\beta|$$

Non-deg. Pert. $\hat{H}_1 \propto \delta$

REMEMBER !! $\Rightarrow$

$$E_n^{(1)} = \langle n^{(0)}|\hat{H}_1|n^{(0)}\rangle$$

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m^{(0)}|\hat{H}_1|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$E_n^{(j+1)} = \langle n^{(j)}|\hat{H}_1|n^{(0)}\rangle$$

$$\hat{O} \doteq \begin{array}{c|cccc} & |0\rangle & |1\rangle & \cdots \\ \langle 0| & O_{00} & O_{01} & \cdots \\ \langle 1| & O_{10} & O_{11} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

$$O_{00} = \langle 0|\hat{O}|0\rangle$$
$$O_{10} = \langle 1|\hat{O}|0\rangle$$
$$\vdots$$

$$|\alpha\rangle \doteq \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix} \doteq \alpha_0|0\rangle + \alpha_1|1\rangle + \cdots$$

$$|\psi\rangle \doteq \psi(x)$$

Spin $\frac{1}{2}$ problem

$$|\psi\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \doteq |S_z = \tfrac{1}{2}\rangle$$

$$\begin{bmatrix} E_1-\lambda & \Delta \\ \Delta^* & E_2-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} : \text{e-vectors}$$

Degenerate perturbation

degeneracy $= 2$, $\{|0^{(0)}\rangle, |1^{(0)}\rangle\}$

Initial basis state

$$\hat{H} \doteq \begin{array}{c|cc} & |0^{(0)}\rangle & |1^{(0)}\rangle \\ \langle 0^{(0)}| & E_1 & \Delta \\ \langle 1^{(0)}| & \Delta^* & E_2 \end{array}$$

in the degenerate sub-space

$\Rightarrow$ Diagonalize it to find $E_0^{(0)} + E_0^{(1)}$ and $E_1^{(0)} + E_1^{(1)}$

$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} \doteq a|0^{(0)}\rangle + b|1^{(0)}\rangle$

There are two sets of (a, b).

$\Rightarrow |0^{(0)}\rangle', |1^{(0)}\rangle'$ Good zeroth order eigenstates

$$\hat{H} \doteq \begin{array}{c|cc} & |0^{(0)}\rangle' & |1^{(0)}\rangle' \\ \langle 0^{(0)}|' & E_0^{(0)}+E_0^{(1)} & 0 \\ \langle 1^{(0)}|' & 0 & E_1^{(0)}+E_1^{(1)} \end{array}$$

$\hat{h}\,|h\rangle = h\,|h\rangle$

- h : discrete

  $\langle h | h' \rangle = \delta_{h,h'}$.

- h : continuous

  $\langle h | h' \rangle = \delta(h - h')$

$$\sum_h |h\rangle\langle h| = 1 \leftarrow \text{discrete}$$

$$\int dh\, |h\rangle\langle h| = 1 \leftarrow \text{continuum}$$

$|\psi\rangle$

Measure $\hat{h}$ on $|\psi\rangle$

$\Rightarrow$ After the measurement

$\Rightarrow |h\rangle\langle h|\psi\rangle$

where h is <u>one</u> of all possible e-values.

Symmetry Ops	infinitesimal	finite
• Time evolution:	$1 - i\dfrac{\hat{H}\,dt}{\hbar}$	$e^{-i\frac{\hat{H}\Delta t}{\hbar}}$ ✓ if $\hat{H}$ is not explicitly dep. on t.
• Translation:	$1 - i\dfrac{\hat{p}\,dx}{\hbar}$	$e^{-i\frac{\hat{p}\Delta x}{\hbar}}$
• Rotation:	$1 - i\dfrac{\hat{L}_\theta\,d\theta}{\hbar}$	$e^{-i\frac{\hat{L}_\theta \Delta\theta}{\hbar}}$

What if it is ?? $e^{-i\frac{\int \hat{H}(t)\,dt}{\hbar}}$ if $[\hat{H}(t), \hat{H}(t')] = 0$

Conservation?

Def $\hat{O}$ is conserved. $\Longleftrightarrow$ $\dfrac{d}{dt}\langle \hat{O} \rangle = 0$ for any state $|\Psi\rangle$

$\langle \Psi | \hat{O} | \Psi \rangle$

$\Longrightarrow$ ① $\hat{H}$ is conserved if $\hat{H}$ is not explicitly dep on t.

② $\hat{O}$ is conserved if $\dfrac{\partial \hat{O}}{\partial t} = 0$ and $\boxed{[\hat{O}, \hat{H}] = 0}$

$\hat{H} = \hat{H}(p, q, t)$ ← in 1D.

$\hat{H}(p_1 \cdots p_N, q_1 \cdots q_N, t)$ ← in N-D.

$\hat{H}$ is not explicitly dep. on t

$\Longleftrightarrow \quad \hat{H} = \hat{H}(p, q)$

$$H = \frac{p^2}{2m} + V(\overset{q}{x})$$

$$= \frac{d}{dt}\left(\frac{p^2}{2m}\right) + V(q)$$

$$\cancel{\hat{B}|A\rangle \overset{?}{=} B|A\rangle}$$

Symmetry principle in QM

① $\hat{A}|A\rangle = A|A\rangle$, $[\hat{A}, \hat{B}] = 0$

$$\Rightarrow \hat{A}\{\hat{B}|A\rangle\} = A\{\hat{B}|A\rangle\}$$

② $\hat{A}$: diagonalizable

$\hat{B}$: ″ , $[\hat{A}, \hat{B}] = 0$

$\Longrightarrow$ $\hat{A}$, $\hat{B}$ are simultaneously diagonalizable.

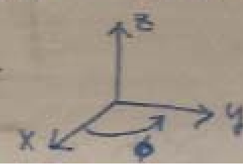
$\hat{A} = \hat{P}_z \ (z \to -z)$

$\hat{B} = \exp\left(-i \frac{\hat{L}_z (\frac{\pi}{2})}{\hbar}\right)$

$[\hat{A}, \hat{B}] = 0$

$|\psi\rangle \doteq C z^2 x$, $\hat{A}|\psi\rangle = 1|\psi\rangle$

$\hat{B}|\psi\rangle \doteq C z^2 y$



$\phi \longrightarrow \phi - \frac{\pi}{2}$

The last slide is worth some more comments. The symmetry principle as stated on the right side is important. The question that was posed during the review session is the following: for given $\hat{A}$ and $\hat{B}$, compatible as stated (i.e., they commute with each other), and for a given eigenstate of $\hat{A}$, $|A\rangle$, is $\hat{B}|A\rangle = B|A\rangle$? And the answer is NO.
The answer would be yes if $|A\rangle$ were a unique eigenstate of $\hat{A}$ for eigenvalue A. If there is any degeneracy, then $\hat{B}|A\rangle$ is not $|A\rangle$ times some number, in general. An example (blue writing in the lower center part) is given to illustrate this point. Here, a z-partity and a 90 degree rotation around the z axis are considered, along with a state $|\psi\rangle \doteq Cz^2x$, where C is a normalization constant. Or, $|\psi\rangle \doteq Cx$ will do, also. While this state remains the even z-parity state after the 90 degree rotation, it does change since $x \to y$, by 90 degree rotation around the z axis (as we saw in a homework problem).
The important point here is that **friends—commuting operators—do not mess with eigenvalues, but eigenvectors *can* be messed up. All we can say is that $\hat{\boldsymbol{A}}(\hat{\boldsymbol{B}}|\boldsymbol{A}\rangle) = \boldsymbol{A}(\hat{\boldsymbol{B}}|\boldsymbol{A}\rangle)$.**

To prove the second principle, the simultaneous diagonalizability, one can proceed as follows. For a given $|A\rangle$, apply $\hat{B}$ to it. We just saw that the result is generally not $|A\rangle$ times some number; thus, $\hat{B}|A\rangle$ in general has some component perpendicular to $|A\rangle$. Do not be alarmed. Simply keep operating $\hat{B}$ on this result, until all resultant states are describable as a linear combination of a certain number of orthonormal vectors. The space spanned by these orthonormal vectors is what we may call the degenerate subspace of $\hat{A}$ for the eigenvalue A. In this subspace (of the total Hilbert space), the $\hat{A}$ operator is simply a number, A, times the identity operator, and so it is always diagonal, no matter what orthonormal vectors you take as the basis set for this subspace. Now, by assumption, $\hat{B}$ is diagonalizable in this subspace. So, do it, and we have diagonalized $\hat{A}$ and $\hat{B}$ simultaneously. Repeat this for all eigenvalues of $\hat{A}$. QED.

Notice that the steps in this proof of the one of the most important theorems of quantum mechanics (principle (2) of the last page) are very similar to what we have been doing with the degenerate perturbation case. In that case, we identify the degenerate subspace of $\hat{H}_0$ and then diagonalize $\hat{H}_1$ (or $\hat{H}_0 + \hat{H}_1$). When we do so, we are simultaneously diagonalizing $\hat{H}_0$ and $\hat{H}_1$ in this degenerate subspace. So, this seems quite similar to what we did in the previous paragraph. However, the identification of $\hat{A}$ and $\hat{B}$ must be carefully done. Often, $\hat{H}_0$ and $\hat{H}_1$ do *not* commute. However, almost always, $\hat{H}_1$ has a certain non-trivial symmetry, which is part of the symmetry of $\hat{H}_0$. Then, one identifies $\hat{H}_0$ as $\hat{A}$ and $\hat{S}$ (the group of compatible symmetry operators for $\hat{H}_1$) as $\hat{B}$.

In general, the important symmetry principle (1) is used in the following way, for diagonalizing the Hamiltonian $\hat{H}$.

1. Identify $\hat{A}$ as the group of compatible symmetry operators for $\hat{H}$, and $\hat{B}$ as $\hat{H}$.
2. Identify degenerate subspaces of $\hat{A}$.
3. By the symmetry principle (1), $\hat{H}$ has no matrix element connecting states in different degenerate subspaces of $\hat{A}$. So, then, our task is to diagonalize $\hat{H}$ in each degenerate subspace of $\hat{A}$. This is how the symmetry principle makes it easier for us to diagonalize $\hat{H}$. Without using the symmetry principle, we would have to diagonalize $\hat{H}$ in the entire Hilbert space. With the symmetry principle, we would need to diagonalize $\hat{H}$ in one degenerate subspace of $\hat{A}$, at a time!

In the above, the notation is a bit sloppy, since we used $\hat{A}$ for a set of compatible operators. This was only for the convenience of notation. Please see Homework 4.1 as an example for using sets of compatible symmetry operators.

In any case, what we must always do is the following. Identify the symmetry of the problem, and write down the most general symmetry eigenstate. Then, this state is an eigenstate of the Hamiltonian. By the most general symmetry eigenstate, we mean a general linear combination of eigenstates with the same symmetry eigenvalue. Note that there will be undetermined coefficients involved to write such a general linear combination. This is how far the symmetry principle can help us. In order to really find the values of the coefficients for the Hamiltonian eigenstates, we must do some work—such work corresponds to diagonalizing the Hamiltonian in the degenerate subspace of the symmetry operator(s).

As an example, consider problem 5 of the practice midterm. The symmetry operator that we can choose is $\hat{R}_z\left(\frac{\pi}{2}\right)$, rotation by 90 degrees around the z axis. (There are other rotations such as rotation by 180 degrees and 275 degrees, but they are all diagonalized, if we diagonalize $\hat{R}_z\left(\frac{\pi}{2}\right)$.) To solve the Hamiltonian using this symmetry, then we must write down the most general eigenstate of $\hat{R}_z\left(\frac{\pi}{2}\right)$, such a state is given by $\sum_{n=-\infty}^{\infty} C_n |m+4n\rangle$ where we define $|m\rangle$ as a state described by the wave function $Ne^{im\theta}$, where N is the normalization constant $1/\sqrt{2\pi}$, θ is the polar angle defined in the xy plane, and m is an integer. (cf. Homework 3.4.)