TISE = time independent Schrödinger equation. SS = stationary state. 1D = one dimension.

$$\hat{H}\left|\Psi(t)\right\rangle = i\hbar\frac{d}{dt}\left|\Psi(t)\right\rangle \qquad \text{Schrödinger equation} \tag{1}$$

$$H(\hat{x},\hat{p},t)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t) \qquad \text{Schrödinger eq., } x\text{-repr, spinless particle in 1D} \tag{2}$$

$$\hat{H}\left|\psi\right\rangle = E\left|\psi\right\rangle \qquad \text{TISE: } \frac{\partial \hat{H}}{\partial t} = 0,\ \text{SS, } \left|\Psi(t)\right\rangle \equiv \exp(-iEt/\hbar)\left|\psi\right\rangle \tag{3}$$

$$\hat{\mathcal{T}}(\Delta x) = \exp\left(-i\frac{\Delta x\,\hat{p}}{\hbar}\right) \qquad \text{translation operator in 1D} \tag{4}$$

$$\hat{\mathcal{R}}(\Delta\theta) = \exp\left(-i\frac{\Delta\theta\,\hat{L}_\theta}{\hbar}\right) \qquad \text{rotation operator} \tag{5}$$

$$\left[\hat{x},\hat{p}\right] = i\hbar \qquad \text{also for other canonical conjugate pairs} \tag{6}$$

$$\left[\hat{\theta},\hat{L}_\theta\right] = i\hbar \qquad \therefore\ \ \hat{L}_\theta \doteq -i\hbar\frac{\partial}{\partial\theta} \tag{7}$$

$$\left[\hat{L}_j,\hat{L}_k\right] = i\hbar\epsilon_{jkl}\hat{L}_l \qquad \text{similarly, if all } L\to S \text{ or if all } L\to J \tag{8}$$

$$\left[\hat{L}_j,\hat{p}_k\right] = i\hbar\epsilon_{jkl}\hat{p}_l \qquad j,k,l = \text{any of } 1,2,3 \text{ (or } x,y,z\text{)} \tag{9}$$

$$\left[\hat{L}_j,\hat{x}_k\right] = i\hbar\epsilon_{jkl}\hat{x}_l \qquad \epsilon_{jkl} = \text{Levi-Civita symbol} \tag{10}$$

$$\left[\hat{L}_j,\hat{L}^2\right] = 0 \qquad \text{similarly, if all } L\to S \text{ or if all } L\to J \tag{11}$$

$$\left[\hat{L}_j,\hat{p}^2\right] = 0 = \left[\hat{L}_j,\hat{r}^2\right] \qquad \hat{p}^2 \equiv \hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2,\quad \hat{r}^2 \equiv \hat{x}^2+\hat{y}^2+\hat{z}^2 \tag{12}$$

$$\left[\hat{L}_j,\hat{S}_k\right] = 0 \qquad \hat{L} \text{ and } \hat{S} \text{ live in orthogonal Hilbert spaces} \tag{13}$$

$$\hat{H} = -\hat{\vec{\mu}}\cdot\hat{\vec{B}} \qquad \text{for magnetic moment } \vec{\mu} \text{ in a } \vec{B} \text{ field} \tag{14}$$

$$\hat{\vec{\mu}} = -\frac{\mu_B}{\hbar}(2\hat{\vec{S}}+\hat{\vec{L}}) \qquad \text{for electron } (g_{spin}\approx 2),\ \mu_B = \frac{e\hbar}{2m_e} = \text{Bohr magneton} \tag{15}$$

$$\left[\hat{A},f(\hat{B})\right] = 0 \ \text{ if } \ \left[\hat{A},\hat{B}\right] = 0 \qquad \text{for any analytic function } f \tag{16}$$

$$f(\hat{B})\left|B\right\rangle = f(B)\left|B\right\rangle \ \text{ if } \ \hat{B}\left|B\right\rangle = B\left|B\right\rangle \qquad \text{for any analytic function } f \tag{17}$$

$$\left\langle\hat{T}\right\rangle = \frac{n}{2}\left\langle\hat{V}\right\rangle \qquad \text{Virial theorem for } \hat{V}\propto\hat{x}^n,\hat{r}^n,\ \text{SS} \tag{18}$$

$$\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\delta(k-k') \qquad \delta(k) = \text{Dirac delta function} \tag{19}$$

$$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \text{Pauli matrices, spin 1/2, } \hat{S}_j \doteq \frac{\hbar}{2}\sigma_j \tag{20}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right) \qquad \text{1D simple harmonic oscillator} \tag{21}$$

$$\hat{p} = \frac{1}{i}\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}-\hat{a}^\dagger\right) \qquad \text{one-dimensional (1D) SHO} \tag{22}$$

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle,\quad \hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \qquad \text{1D SHO, “ladder” operators} \tag{23}$$

Please organize your solutions as neatly as you can.
Please start by writing down your name on your solution sheet.
Show **sufficient minimal derivation** for all your answers.

Each of problems 1-5 is worth 50 points.
You are required to do **only four problems out of problems 1-5.**
However, if you do all five, then you may earn extra credit.
Problems 6,7 are required, while problems 8,9,10 are for extra credit.

Good luck!

Problem 1 A simple harmonic oscillator is subject to a small perturbation $\hat{H}_1 \doteq \alpha x^3$. Does the ground state energy increase or decrease due to this perturbation? Briefly explain your answer.

Problem 2 Find eigenstates of $\hat{p} \doteq -i\hbar\frac{\partial}{\partial x}$. Find normalization constants to satisfy the condition $\langle p_1 | p_2 \rangle = \delta(p_1 - p_2)$, where p_1 and p_2 are $\hat{p}$ eigenvalues, and $|p_1\rangle$ and $|p_2\rangle$ are corresponding eigenstates.

Problem 3 Consider a particle in one dimension. Suppose that its potential energy is given by the sum of two identical square potential wells ($\hat{V}_R$ and $\hat{V}_L$) of a certain finite depth and a certain finite width. The depth and the width are such that the square well supports at least one bound state.

Initially, the two wells are so far away from each other, so that they can be considered completely separate: i.e., if the particle is found in one well, then the probability that it will tunnel to the other well is zero for all practical purposes. In particular, the ground state is degenerate: it is given by $|R\rangle$ or $|L\rangle$, where $|R\rangle$ is the state in which the particle is in the ground state of the right well, and $|L\rangle$ is the state in which the particle is in the ground state of the left well.

Then, the two wells are brought close to each other so that the particle, if found in one well, can now tunnel into the other well. The distance between the two wells is close enough so that the tunneling probability considered above is now non-zero, but is far enough so that the tunneling probability is very small. We also assume (for convenience, which is ultimately justified as it does not ruin the main thread of physics)

$$\langle R | L \rangle = 0$$

Under these assumptions, the perturbed ground state can be approximated as a linear combination of $|R\rangle$ and $|L\rangle$. Which one corresponds to the perturbed ground state: $\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$ or $\frac{1}{\sqrt{2}}(|R\rangle - |L\rangle)$? Show your derivation.

[Hints: It might help to write $\hat{H} = \hat{T} + \hat{V}_R + \hat{V}_L = \hat{H}_R + \hat{V}_L = \hat{H}_L + \hat{V}_R$, and note that $|R\rangle$ is an eigenstate of $\hat{H}_R$ and $|L\rangle$ is an eigenstate of $\hat{H}_L$. This problem does *not* require the knowledge of the ground state wave function for a single well. It does, however, require the knowledge of some general (symmetry-related) features of the ground state wave function for a single well.]

Problem 4 Is the following statement true? If true, prove it. If false, provide a counter example.

If two operators $\hat{A}$ and $\hat{B}$ are compatible, then any eigenstate of $\hat{A}$ is automatically an eigenstate of $\hat{B}$.

Problem 5 Consider an electron in two dimensions. It is subject to a combination of Coulomb potentials due to five charges, fixed in space: a positive charge at origin, and four equal negative charges at $(x, y) = (\pm a, 0)$ and $(0, \pm a)$. Identify any translational or rotational symmetry of this problem, *continuous or discrete*, and find the most general form of energy eigenstate implied by that symmetry alone (i.e. find the most general form of the eigenstate of the symmetry identified). Note that there is only one rotational axis (z) in this problem, as the motion is confined to a plane. [Hint: a discrete translation/rotation symmetry means that the Hamiltonian is invariant under translation/rotation by certain discrete values only.]

Problem 6 (100 points) A spin 1/2 particle is prepared so that its state is given by $|z\uparrow\rangle$ at time $t = 0$, where $|z\uparrow\rangle$ means the state in which $S_z = \hbar/2$. Ignore the orbital angular momentum of the particle, e.g. by assuming an s state. At $t = 0$, a uniform magnetic field, $\vec{B}_0$, is turned on suddenly. For $t \geq 0$, the field is constant in time. At some later time $t > 0$, the spin angular momentum $\hat{S}_x$ is measured. Find the probability $P_{x,\uparrow}(t)$ to measure $S_x = \hbar/2$ and the probability $P_{x,\downarrow}(t)$ to measure $S_x = -\hbar/2$, at that time, for each of the following two scenarios. Show your derivation clearly for each part. [Hint: the Hamiltonian is given by $\hat{H} = \frac{2\mu_B}{\hbar}\vec{B}_0 \cdot \hat{\vec{S}}$.]

(a) $\vec{B}_0 = B_0\,\vec{e}_3$, where $\vec{e}_3$ is the unit vector along the z direction.

(b) $\vec{B}_0 = B_0\,\vec{e}_1$, where $\vec{e}_1$ is the unit vector along the x direction.

Problem 7 (100 points) A two dimensional simple Harmonic oscillator defined by the potential

$$V(x, y) = \frac{k}{2}(x^2 + y^2)$$

is subjected to a small perturbation

$$\hat{H}_1 \doteq \epsilon k(x^2 + xy)$$

where $|\epsilon| \ll 1$. Find the first order correction in energy for the first three states corresponding to the ground state and the first excited state. Your answers must expressed in terms of ϵ and $\hbar\omega$ only, where $\omega = \sqrt{k/m}$. If other symbols, such as k, appear in your answers, convert them.

Problem 8 (Extra credit; 100 points) Consider a three dimensional simple harmonic oscillator problem for an electron. The potential energy for the electron, $\hat{V} \doteq V(x,y,z)$, is given by

$$V(x,y,z) = \frac{k}{2}(x^2 + y^2 + z^2)$$

(a) Find all energy eigenstates and all energy eigenvalues.

(b) Find the first order correction of the ground state energy due to the perturbation

$$\hat{H}_1 \doteq f(r)\hat{\vec{L}} \cdot \hat{\vec{S}}$$

where

$$\langle f(\hat{r}) \rangle$$

for the ground state can be left as is (or as a constant symbol). Note that the ground state wave function of a *one-dimensional* simple harmonic oscillator is given by $N \exp(-x^2/2)$ where N is a normalization constant. You are *not* advised/required to evaluate N!

Problem 9 (Extra credit; 50 points) Find the second order correction to the ground state energy for problem 7. Your answers must expressed in terms of ϵ and $\hbar\omega$ only, where $\omega = \sqrt{k/m}$. If other symbols, such as k, appear in your answers, convert them.

Problem 10 (Extra credit; 50 points) Find the first excited states and their degeneracy (including the spin degeneracy) for problem 8, part ((a)). Find the first order correction to their energy by the perturbation in part ((b)) of problem 8. Note that the wave function for the first excited state of a *one-dimensional* simple harmonic oscillator is given by $Nx \exp(-x^2/2)$ where N is a normalization constant. You are *not* advised/required to evaluate N! Any radial integral that arises can be left as is (or as a constant symbol).

Hint for problems 7, 8: the unperturbed problem separates into an independent simple harmonic oscillator problem for each degree of freedom (x, y, or z).