Not due, but must do.
If you submit high quality solutions for any problem(s), and if your solutions turn out to be high quality indeed, then you will get very significant extra credit points.

Problem 1 Optical theorem and shadow scattering.

(a) From Eqs. 17.24 and 17.27, prove that

$$\sigma = \frac{4\pi}{k}\,\mathrm{Im}\, f(\theta = 0)$$

This is the so-called **optical theorem**.

(b) Verify the optical theorem for the hard sphere problem in the low energy limit (Eq. 17.34). [Hint: Eq. 17.31 is clearly not enough for evaluating the imaginary part. The next order correction in kR is necessary.]

(c) Consider the same hard sphere problem, but in the high energy limit $kR \to \infty$. Show that the phase shift is given by $\delta_l \approx \frac{l\pi}{2} - kR$ in this limit.

(d) Noting that $f_l(k) = (S_l(k) - 1)/(2ik)$ with $S_l(k) = e^{2i\delta_l}$, one can split the $f(\theta)$ into two parts, the reflection part and the "shadow scattering" part.

$$\begin{aligned} f(\theta) &= f_r(\theta) + f_s(\theta) \\ f_r(\theta) &= \sum_{l=0}^{\infty} (2l+1)\,\frac{e^{2i\delta_l}}{2ik}\,P_l(\cos\theta) \\ f_s(\theta) &= \sum_{l=0}^{\infty} (2l+1)\,\frac{i}{2k}\,P_l(\cos\theta) \end{aligned}$$

Now, in the $kR \to \infty$ limit, we like to evaluate these sums, at $\theta = 0$, to apply the optical theorem. Clearly, the sum for f_s is purely imaginary, and *is divergent* for a *fixed* value of k for $\theta = 0$: $f_s(\theta = 0) = \frac{i}{2k}\sum_l (2l+1)$. However, in the limit of $kR \to \infty$, one can make the following semi-classical arguments[1].

(a) The angular momentum is given by $mvb = \hbar kb$, with $b < R$. And so, $l = kb$ and we expect that $l_{max} = kR$.

(b) Then, the sum over l can be converted to an integral with the following substitutions $\sum_l \to k\int_0^R db$ and $l \to kb$.

By following this prescription, show the following

$$\mathrm{Im}\, f_s(\theta = 0) \approx \frac{kR^2}{2} \qquad \text{leading order}$$

$$\mathrm{Im}\, f_r(\theta = 0) \approx \frac{1}{k}O(kR) \qquad \text{negligible compared to } \mathrm{Im}\, f_s(\theta = 0)$$

[1] While these arguments are physically plausible, the mathematical procedures dictated by them *can not* be derived naturally in the current formalism. This is because the simple plane wave formalism that we are using here is not good enough for taking the theory all the way to the classical limit.

Using these results and the above optical theorem, show that the total cross section is given by

$$\sigma = 2\pi R^2$$

(e) From the previous part, it is clear that the f_s part is important for the cross section. Now, we take a different (more brute force) approach to calculate the total cross section. Taking f_r and f_s, *one at a time*, show that

$$\sigma_r \equiv \int d\Omega\, |f_r(\theta)|^2 \approx \pi R^2$$

and

$$\sigma_s \equiv \int d\Omega\, |f_s(\theta)|^2 \approx \pi R^2$$

using the above trick to convert an l sum to an integral using semi-classical arguments. (Note that, obviously, one *cannot* use the optical theorem for f_r and f_s separately. Here, one has to calculate the integral over the solid angle, directly.) These results, when compared with the result of part (d), imply that $\sigma = \sigma_r + \sigma_s$ and that the interference term $\int d\Omega\,(f_r f_s^* + f_r^* f_s) = 0$, which one can prove explicitly, if one so desires.

[Note: Why is the term f_s called the *shadow scattering* term? The reason can be most clearly seen if we look at what is happening in part (d). In the forward scattering direction, the scattering amplitude is such that f_s is dominant over f_r and is i times a positive number, in addition to being purely imaginary. Now, note that what we call f_r is in fact, $2ik\sum_l (2l+1) S_l P_l$, i.e., it represents the amplitude of the *total* outgoing wave $2ik f_r e^{ikr}/r$, not just the scattered part $(f(\theta))$. Noting now that $f_r(\theta) = f(\theta) - f_s(\theta)$, we see that the total outgoing spherical wave component is zero in the forward scattering direction (thus a *shadow*—note that the scattering cannot do anything on the incoming spherical wave), i.e., $f_r(\theta = 0) = 0$, because $f(\theta = 0)$ and $-f_s(\theta = 0)$ cancel each other. This makes sense, since $-f_s(\theta)$ is simply the outgoing wave component of the original plane wave (cf. Eq. 17.18).]

Problem 2 Scatteing length. Let us consider a general potential, $V(r)$, with the following conditions:

$$\begin{aligned} V(r) &= 0 && \text{if } r > R \\ |V(r)| &< \infty && \text{if } r < R \end{aligned}$$

So, R defines the finite range of the potential. Consider the scattering of a very low energy $k \approx 0$ beam. Then, it suffices to consider the s wave ($l = 0$) scattering alone[2]. Since the exact form of $V(r)$ is not specified, the phase shift δ_0 is left as

[2]Strictly speaking, we have shown this only for a hard sphere potential, in Eq. 17.31, but this argument can be generalized to any finite ranged potential.

a mere symbol. It and k determine the the problem completely. Let us assume that k is small enough

$$kR \ll 1$$

so that the conditions

$$r > R, \quad kr \ll 1$$

are valid for r, greater than R, but not too large. We consider such r values only.

(a) Show that the wave function for such r is given by

$$\psi(r,\theta) \approx A(r-a)/r$$

with

$$a = -\delta_0/k$$

Here, a is the so-called **scattering length**. For an attractive potential, $|a|$ can be much greater than R, while for weak potential, $|a|$ is on the order of R or smaller. Show that for self-consistency

$$k|a| \ll 1$$

is required. [Hints: Eq. 17.28, and the small x limits of j_0 and n_0 (Eqs. 17.6 and 17.7) must be considered. Assume that δ_0 is small, consistent with the assumption that only the s wave consideration is sufficient.]

(b) Show that the total cross section σ is given by

$$\sigma = 4\pi a^2$$

(c) Let us assume that the potential is attractive and there is a shallow s-wave *bound state* for which the bound state energy is given by

$$E_b = -\frac{\hbar^2\kappa^2}{2m} \approx 0$$

Show that the wave function for such a bound state must be given as, for $r > R$,

$$\psi(r,\theta) = Ce^{-\kappa r}/r$$

[Hint: consider the radial equation, Eq. 17.4.]

(d) Continuing our consideration of a shalllow bound state, let us assume that κ is small enough,

$$\kappa R \ll 1$$

so that for $r > R$, but not too large r, we can satisfy $\kappa r \ll 1$. Show that for such r the wave function of the shallow bound state can be written in the form

$$\psi(r,\theta) \approx B(r-b)/r$$

where B is a normalization constant. Find b in terms of κ.

(e) Let us consider the radial equations (Eq. 17.3) when $r < R$, for the two cases, the low energy scatteirng state (small k) and the low energy bound state (small κ). These two radial equations are approximately the same, as E can be ignored for $r < R$ (where the potential energy is now substantial and dominates over E). The *full* solution to the radial equation is determined by connecting the wave function $u(r)$ across the boundary $r = R$. By considering the continuity of $u(r)$ and $u'(r)$ at R, show that

$$b = a$$
$$E_b = -\frac{\hbar^2}{2ma^2}$$

on condition that

$$a \gg R.$$

This remarkable result shows that the energy of a shallow bound state may be inferred from a low energy scattering experiment!

Problem 3 Problem T11.4. Low energy scattering on a delta function potential.

Problem 4 Problem T11.12. Total cross section of Yukawa potential. Born approximation.

Problem 5 Problem T11.13. Born approximation. Delta function potential.

Problem 6 Problem T10.6. Berry phase for the angular momentum rotation. For full credit, do this problem for a general angular momentum j. For the general case the following **Baker-Hausdorff formula** will be useful, along with our knowledge of the angular momentum rotation operator.

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \ldots$$