Due Dec 4, Wednesday

Problem 1 (30 points) Prove, using the *classical mechanics* (i.e. Hamilton's equations of motion) that Eq. 14.6 follows from the Hamiltonian Eq. 14.1.

Problem 2 (50 points; **Landau levels**) Consider the motion of an electron in a two dimensional plane with a time-independent uniform magnetic field pointing along the z direction, $\vec{B} = B_0 \vec{e}_z$ ($\vec{e}_z$ is the unit vector along the z direction), perpendicular to the plane.

(a) (*Classical E&M*) Clearly, the scalar potential ϕ can be taken as zero, for this problem. Find the vector potential $\vec{A}$. There may be many ways to do this. I recommend the way like in pages 3 and 4 of LN 14, using the cylindrical symmetry of the problem.

(b) (*Newtonian mechanics*) *Briefly*, but sufficiently, explain *why* the electron undergoes a *uniform circular motion* and how its orbit size, R, is determined by v (speed), and B_0 while the angular frequency ω_c is determined by B_0 only (plus other fundamental constants of nature, of course). Find R and ω_c. Also, specify the direction of the circular motion.

(c) (*Semi-classical physics*) Use the Bohr-Sommerfeld quantization rule

$$\oint p_\theta d\theta = (n+\gamma)h \qquad n = 0, 1, 2, \ldots \quad (n \gg 1)$$

(p_θ is the angular momentum associated with θ. Maybe a better notation for p_θ is L_θ. In any case, $p_\theta = L_\theta = (\vec{r} \times \vec{p})_z$.) to obtain (i) the quantized energy values ($E_n = \frac{1}{2} m_e v^2$), and (ii) the quantized magnetic flux ($\int d\vec{\sigma} \cdot \vec{B}$) enclosed by the semi-classical orbit, whose increment must be the *flux quantum*

$$\Phi_0 = \frac{h}{e} = 4.14 \times 10^{-15} \text{ T-m}^2.$$

(d) (*Quantum mechanics – Landau levels*) Use the vector potential that you obtained above, to solve the problem quantum mechanically (cf. Eq. 14.13). This problem is exactly solvable. Show that the problem maps to a one dimensional simple harmonic oscillator, and obtain the quantized energy in terms of the $\hbar\omega_c$. [Hint: If you are doing the problem with a vector potential with the cylindrical symmetry, then examine the commutator $\left[\hat{p}_y + \frac{e}{2} B_0 \hat{x}, \frac{1}{eB_0}\left(\hat{p}_x - \frac{e}{2} B_0 \hat{y}\right)\right]$.]

(e) (*Flux/energy quantization*) Show that for a certain value of γ, which you must identify, the semi-classical result for E_n becomes the exact quantum result.

Problem 3 (20 points; **Dipole approximation**) Prove Eq. 15.5 and then, using that result, prove Eq. 15.7.

Problem 4 (30 points; **Stimulated emission**) Derive Eqs. 15.24 and 15.25 following the instruction given right before them.

Problem 5 (30 points) Problem T9.14

Problem 6 (40 points; extra credit) Problem T9.11

Problem 7 (40 points; extra credit) Given the quantum operator Eq. 14.37 for photon, calculate the free photon Hamiltonian

$$\hat{H} = \int_{\mathcal{V}} d^3\vec{r}\, u(\vec{r}, t)$$

where the energy density u is given by Eq. 14.31 and $\vec{E}$ and $\vec{B}$ are given by Eqs. 14.29 and 14.30, but now with the quantized A_0 operator and its "complex conjugate" (i.e. Hermitian conjugate) as given by Eq. 14.37. This Hamiltonian corresponds to the energy of only one mode of photon with definite values of $\vec{k}$ and $\vec{\varepsilon}$. Show that the result is given as

$$\hat{H} = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)\hbar\omega \tag{1}$$

where the subscripts $(\vec{k}, \vec{\varepsilon})$ for $\hat{a}$ and $\hat{a}^\dagger$ are omitted for brevity (cf. Eq. 14.37). Note that time does not appear in this equation: i.e., you must show that the integral $\int_{\mathcal{V}} d^3\vec{r}\, u(\vec{r}, t)$ is time independent. Consider the volume $\mathcal{V}$ as a cube of equal sides L ($L^3 = \mathcal{V}$), and apply the periodic boundary condition (cf. page 6 of LN 15).

(*Note*) The total Hamiltonian for all free photon modes is then given by

$$\hat{H} = \sum_{\vec{k}, \vec{\varepsilon}} \left(\hat{a}^\dagger_{\vec{k}, \vec{\varepsilon}} \hat{a}_{\vec{k}, \vec{\varepsilon}} + \frac{1}{2}\right)\hbar\omega_{\vec{k}}$$