Due Oct. 8, Tuesday

Problem 1 (10 points) Consider two operators $\hat{O}_1$ and $\hat{O}_2$. We say that they are equal to each other if all matrix elements for them are equal, i.e., if

$$\left\langle \alpha \left| \hat{O}_1 \right| \beta \right\rangle = \left\langle \alpha \left| \hat{O}_2 \right| \beta \right\rangle$$

for any states $|\alpha\rangle$ and $|\beta\rangle$.

(a) Given the above definition, prove the following: $\hat{O}_1$ and $\hat{O}_2$ are equal *if and only if*

$$\left\langle \alpha \left| \hat{O}_1 \right| \alpha \right\rangle = \left\langle \alpha \left| \hat{O}_2 \right| \alpha \right\rangle$$

for any state $|\alpha\rangle$. [Hint: consider $|\alpha\rangle = |\alpha_1\rangle + |\alpha_2\rangle$ and then $|\alpha\rangle = |\alpha_1\rangle + i\,|\alpha_2\rangle$ for given arbitrary states $|\alpha_1\rangle$ and $|\alpha_2\rangle$.]

(b) Suppose that an observable, $\hat{\xi}$, has a mixed eigenvalue spectrum, some discrete eigenvalues $\{\xi_i\}$ and some continuous $\{\xi\}$. Its eigenvectors are (Dirac-)normalized.

$$\begin{aligned} \langle \xi_i \,|\, \xi_j \rangle &= \delta_{i,j} \\ \langle \xi \,|\, \xi' \rangle &= \delta(\xi - \xi') \\ \langle \xi \,|\, \xi_i \rangle &= 0 \end{aligned}$$

Consider an arbitrary physical state $|\alpha\rangle$. Show, using the result of (a), that the resolution of identity

$$\sum_i |\xi_i\rangle \langle \xi_i| + \int d\xi \, |\xi\rangle \langle \xi| \; = \; 1$$

can be proved readily using the condition that the total probability of all possible outcomes of measuring $\hat{\xi}$ is unity for any physical state.

Problem 2 (15 points) Let us consider a particle moving in a one dimensional world, which we take to be a circle of circumference L. We require that any physical wave function $\Psi(x,t)$ be a single-valued function, where x is the spatial coordinate along the circumference of the circle and t is the time coordinate. Thus, for the spatial part, we require

$$\Psi(x+L,t) \; = \; \Psi(x,t) \qquad \text{for any } x, t.$$

This condition is referred to as the "peridic boundary condition."

(a) Let us take a plane wave function (i.e., a momentum eigenstate), Ae^{ikx}. Show that k is quantized, by the above periodic boundary condition, and find those quantized values.

(b) Take $|k\rangle \doteq Ae^{ikx}$. Prove that with an appropriate constant A, we can indeed satisfy (cf., Eq. 1.9)

$$\langle k' | k \rangle = \delta_{k,k'}.$$

As you prove the above identity, you must also find A.

(c) Then, show that

$$\sum_{k'} \langle k' | k \rangle = 1,$$

where the sum is over all allowed values of wave vector k'.

(d) In the limit of an infinitely large system, $L \to \infty$, we can convert the sum in the previous part to an integral ($\sum_{k'} \to \int_{-\infty}^{\infty} dk'/\Delta k$, where Δk is the quantum of k that you found in the first part). By carrying out this conversion, show that you can prove [1] the well-known identity Eq. 2.22, proving that, in the $L \to \infty$ limit, the momentum eigenstates can be normalized as $|k\rangle \doteq \frac{1}{\sqrt{2\pi}} e^{ikx}$ (cf., Eq. 2.23), so that

$$\langle k' | k \rangle = \delta(k - k').$$

(e) Using the orthonormality $\langle k' | k \rangle = \delta_{k,k'}$ that we demonstrated above, one can prove that

$$\sum_{k} |k\rangle \langle k| = 1,$$

in the case of a finite L. Show that this resolution of identity for the finite L case is converted to the expected resolution of identity for the infinite L case

$$\int dk \, |k\rangle \langle k| = 1, \qquad (1)$$

if the normalization of $|k\rangle$ is taken to be that in part (d), rather than that in part (b).

(f) Consider a particle in a "hyper-cube" of volume L^D, where $D \geq 1$ is the spatial dimension, and L is the linear dimension of the hyper-cube. We can apply the same boundary condition in each spatial coordinate, thereby considering a "hyper-torus" as our space[2]. Write down the properly normalized plane wave function (i) in the case of finite L and (ii) in the case of $L \to \infty$.

[1] This is one way to prove this famous integral. Another way is to use the Contour integral technique. See `http://griffin.ucsc.edu/teaching/08Q2-139A/download/Math%20Note.pdf` (page 5).

[2] As strange as it may sound, this is the most convenient way to consider particles in a finite size space!

Problem 3 (15 points) Equations (2.20) through (2.25) have been derived assuming that the guitar string is infinite in length ($-\infty$ to ∞). However, for a real guitar string, we must restrict $x = 0$ to L, and also apply the boundary condition that any wave of the string must vanish at $x = 0$ and L. Accordingly, values of allowed k values are discrete.

(a) Find these discrete k values, and equations for these discrete k-basis (sine wave), corresponding to equations (2.20) through (2.25).

(b) What would be a QM problem that is exactly equivalent in math to this guitar string problem?

Problem 4 (10 points) Fill in the steps for proving the conservation principles stated in the box of page 20 of LN 2, starting from the two equations given immediately prior to the box.

Problem 5 (15 points) It is known that a spin 1/2 state $|\chi\rangle$, if rotated 360 degrees, becomes $-|\chi\rangle$.

(a) Prove this fact, in as simple a way as you can, starting from our knowledge of the spin rotation operator (Section 2.1.4).

(b) Find the explicit matrix (in the usual $\hat{S}_z$ representation) for the spin rotation operator for rotating a half spin by an arbitrary amount θ around the z axis.

Problem 6 (20 points) Consider a spin 1/2 particle. At time $t = 0$, we measure $\hat{S}_y$ and find a value of $\hbar/2$ for its eigenvalue. Immediately after this measurement, we apply a uniform time *dependent* magnetic field parallel to the z-axis. The B-field is chosen such that the Hamiltonian is:

$$\hat{H}(t) = A(t)\hat{S}_z$$

where

$$A(t) \;=\; \omega_0 t/T \quad \text{for } 0 \le t \le T, \text{ and } \quad 0 \quad \text{for } t > T.$$

(a) Set up and solve the (time-dependent) Schrödinger equation for the spin-1/2 wave function. Show that at time t, the corresponding state is given by:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{i\theta(t)}\,|\alpha\rangle + ie^{-i\theta(t)}\,|\beta\rangle\right]$$

where $|\alpha\rangle$ and $|\beta\rangle$ are eigenstates of $\hat{S}_z$ with eigenvalues $\pm\hbar/2$, respectively, and $\theta(t)$ is a real function of time that you should determine explicitly.

(b) At a time $t > T$, we measure $\hat{S}_y$ again. What are the possible results of this measurement and with what probabilities? Find a relation between ω_0 and T such that the measurement of $\hat{S}_y$ yields a unique result.

For the last two problems, Pauli matrices may be helpful: $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.