How to take this exam

1. **This exam must be taken alone.** You should not consult with anyone about the exam, through any on-line or off-line communication, with one exception—you can ask clarifying questions or procedural questions to me, by email, by posting question on the forum, in person, or by phone. However, you can learn from any static, or quasi-static, materials on-line or off-line—but your solution must explain whatever you may have learned basing off of our course materials (see next).

2. **Show sufficient minimal derivation or explanation for all your answers.** You can refer to equations in lecture notes or the textbook. However, anything going beyond them must be derived.

3. **This exam is due 5 PM sharp, December 11, Wednesday, 2013.** Your solutions must be handed to me in person. Please do *not* leave them in my mailbox—they will not count. I will be in my office all day on Dec 11, and the same applies in general to any weekdays before it.

4. **I am available to answer any questions that you may have about the exam.** Those questions that try to clarify exam questions will be answered. The same applies to those questions that ask about procedural aspects of the exam. Questions asking about how to do problems will not be answered, as a rule. However, for certain problems, it may be that some additional hints will be offered based on questions received.

5. **Any hints, corrections, or clarifications, will be posted on the front page of the course web site, and email notifications will be sent out to all students.** I am generally available in my office—in particular, our office hours apply on Dec 6 and Dec 9. However, you can always reach me by email, or by posting questions on the forum.

6. **Good organization of your solutions is very important for this exam.** Please be very conscientious about it. Consider this exam as your final report, in which you try to show your best of the best. If your solutions are not easily readable, you may not earn points commensurate with your knowledge. Both neat writing and logical presentation are required.

7. **This exam consists of five required problems and one optional fun problem.**

Good luck!

Problem 1 (50 points) Write down the Fermi's golden rule expression for the transition rate when a sinusoidally time-dependent perturbation with frequency ω is applied, including the time-independent case ($\omega = 0$). We consider both the initial state and the final state as unique. Then, discuss the physical meaning of the δ function term that involves frequencies. Your discussion must address the question why that delta function term, and indeed the Fermi's gold rule, arises only in the long time limit, and what the "long time limit" means in precise terms.

Problem 2 (150 points) Consider the following potential energy for a particle moving in one dimension.

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2(x+a)^2 & x < -a \\ 0 & -a < x < a \\ \frac{1}{2}m\omega^2(x-a)^2 & x > a \end{cases}$$

where m is the mass of the particle, $a > 0$ is a length scale constant, and $\omega > 0$ is an angular frequency scale constant. As usual, the kinetic energy is given by the non-relativistic kinetic energy, $p^2/(2m)$.

(a) Use the Bohr-Sommerfeld quantization rule to calculate quantized energy values.

(b) Verify that in the two limits, "$a \to 0$" and "$\omega \to \infty$," your solutions converge to the values that correspond to the simple harmonic oscillator limit ($a \to 0$) and an appropriate limit, which you must identify, respectively. It is also your job to identify exactly what those two limits are in the following sense: they must be expressed clearly as $\lambda \to 0$ and $\lambda \to \infty$ where λ is a *dimensionless* parameter.

(c) What are the values of γ that you can use to make the above two limit cases exact even in the quantum limit (small n)? You are free to choose different γ values for the two limiting cases.

Problem 3 (200 points) In vacuum, there is an electron and the potential energy of the electron is given by the following simple harmonic potential energy

$$V(\vec{r}) = \frac{1}{2}k(x^2 + y^2 + z^2),$$

while its kinetic energy is given by the usual non-relativistic kinetic energy. The value of the spring constant (k) is such that

$$\hbar\omega = \hbar\sqrt{\frac{k}{m}} = 10 \text{ eV}.$$

Initially, the electron is in the first excited state. Calculate the lifetime, in unit of second, of this state. If the first excited state is not unique, then do the calculation for each possible state.

Problem 4 (200 points) Consider a non-relativistic one dimensional motion of a particle with mass m with the following double delta function potential energy:

$$V(x) = -\alpha\Big[\delta(x+a)+\delta(x-a)\Big]$$

where $\alpha > 0$ and $a > 0$. As usual, the kinetic energy is given by the non-relativistic kinetic energy, $p^2/(2m)$.

(a) Solve the time-independent Schrödinger equation for any possible bound states to obtain possible negative energy eigenvalues. The equations for the energy eigenvalues may not be analytically solvable. Leave them as implicit equations for $\kappa = \sqrt{-2mE}/\hbar$, if this is the case, and use a graphical method to discuss your solution. [Note: (i) you are advised to use the symmetry argument very early on in your solution, and (ii) other than the symmetry argument, you can consider this more like a 139A problem.]

(b) How many bound state solutions exist? Does your answer depend on the value of a? Explain. [Hint: you must consider limiting cases for a.]

(c) In the limit of $a \to \infty$, show that you recover the value of the energy, which is well-known for the single delta function potential well (cf., textbook or one of our homework problems).

(d) In the limit of $a \to \infty$, evaluate the leading order correction to the value that you obtained in the previous part, for each bound state solution.

(e) The result of the previous part can be understood from the perturbation theory point of view. Which perturbation theory is relevant here, the non-degenerate theory or the degenerate theory? Explain, based both on your intuitive expectation as well as on your answers to the previous part.

Problem 5 (200 points) Consider a Hamiltonian for an electron in two spatial dimensions

$$\hat{H} = \frac{2v_0}{\hbar}\hat{\vec{p}}\cdot\hat{\vec{S}} = \frac{2v_0}{\hbar}\left(\hat{p}_x\hat{S}_x + \hat{p}_y\hat{S}_y\right)$$

where $\hat{\vec{S}}$ is the spin 1/2 operator, $\hat{\vec{p}}$ is the momentum operator, and $v_0 > 0$ is a constant representing a velocity scale.

(a) Does this Hamiltonian conserve the linear momentum $\hat{p}_x$? How about $\hat{p}_y$? Explain.

(b) Does this Hamiltonian conserve $\hat{S}$, the magnitude of the spin angular momentum? Explain.

(c) Does this Hamiltonian conserve $\hat{S}_x$? How about $\hat{S}_y$ or $\hat{S}_z$? Explain.

(d) Find all eigenstates of $\hat{H}$, making use of what you have found in previous parts. [Hint: you probably need to solve a 2×2 matrix problem, with *complex* off-diagonal elements.]

(e) For the energy eigenstates, show that $\left\langle \hat{\vec{p}} \right\rangle$ and $\left\langle \hat{\vec{S}} \right\rangle$ are either parallel or anti-parallel to each other, by *explicitly analyzing* the nature of the wave function that you have obtained in the previous part[1].

(f) [Extra credit; 50 points] This type of Hamiltonian occurs in a real material, and the backward elastic scattering from $\vec{k}$ to $-\vec{k}$ is mainly responsible for the temperature dependence of the resistivity. Explain, then, why the Berry's phase argument (cf. HW 8.6) implies that the temperature dependence of the resistivity is very small if we make a simple assumption that the backward elastic scattering is totally responsible for the temperature dependence of the resistivity and the backward scattering processes involving different ways of rotating the spin have the exactly the same[2] quantum amplitude except for the Berry's phase.

Problem 6 (extra credit) [This is a fun question. If you give a *short* answer to this using rigorous quantum arguments, then you may get extra credits commensurate with the quality of your answer. Your conclusion may not be that important compared to the logical discussion of your knowledge of quantum mechanics.] We learn from history that Zeno's paradoxes have intrigued human minds for a long time. The so-called third paradox says something like this:

Consider an arrow flying with a uniform velocity. At any fixed time, the arrow, or its center of mass (if you want to be precise), occupies one point in space. Therefore, at that instant, it is motionless. Thus, at any *instant, the arrow is motionless. The arrow cannot be flying.*

Discuss, using your quantum mechanics knowledge, whether there is anything wrong with this argument (or not).

[1] An argument based on the eigenvalues alone is also possible. But, here, you are *required* to provide your arguments based on the properties of the *eigenstates* alone. In reality, this would be a way to double-check that your eigenstates are correct.

[2] This would be the case if the interaction that causes the scattering is a spin-independent interaction.