Please provide your solutions on a separate sheet of paper provided.
Write your name down first, on that sheet!
You can keep this sheet.

1 problem, 15 minutes.

Problem 1 Consider a particle in one dimension, moving in a periodic potential

$$\hat{H} = \hat{T} + \hat{V}$$
$$\hat{T} = \frac{\hat{p}^2}{2m}$$
$$V(x+a) = V(x)$$

Therefore, the system is invariant under discrete translation ("lattice translation") whose operator is given by

$$\hat{\mathcal{T}}(na) = \exp\left(-i\frac{na\,\hat{p}}{\hbar}\right) \qquad n = \text{an arbitrary integer}$$

(a) Show that $\exp(ikx)$ is an eigenfunction of $\hat{\mathcal{T}}(na)$, and find the corresponding eigenvalue.

(b) Write down *the most general form* of eigenfunction for $\hat{\mathcal{T}}(na)$, which contains the plane wave $\exp(ikx)$. This is guaranteed to be an energy eigenstate by symmetry principles.

(c) Your solution in part (b) must involve a sum over various plane waves. Now, assume that $\hat{V}$ is a small perturbation, in addition. Then, the zeroth order states can be taken as plane wave states (eigenstates for $\hat{T}$): $|k\rangle \doteq A\exp(ikx)$, where A is a normalization constant (no need to bother to evaluate A in this problem). Due to off-diagonal matrix elements of the perturbation, i.e. $\langle k|\hat{V}|k'\rangle$ with $k \neq k'$, plane waves get mixed in a true eigenstate. The general form of a true eigenstate was already found in part (b), a form that a perturbative solution also must conform to, required by symmetry principles.
Now, here is a simple question. Take a zeroth order state as the plane wave state corresponding to $k = \pi/a$. Would an *off-diagonal* matrix element of $\hat{V}$ (assume any off-diagonal matrix element to be non-zero) have a first-order perturbation effect or a second order perturbation effect on *this* plane wave state? [Hint: you only need to investigate whether you have a non-degenerate perturbation or a degenerate perturbation, a task made simple, hopefully, by your answer of part (b).]

Your name: