

Due Nov. 29, Thursday (extra office hour on Nov. 28)

Problem 1 (30 points) Prove, using the *classical mechanics* (i.e. Hamilton's (canonical) equations of motion) that Eq. 14.6 follows from the Hamiltonian Eq. 14.1.

Problem 2 (50 points; **Landau levels**) Consider the motion of an electron in a two dimensional plane with a time-independent uniform magnetic field pointing along the z direction, $\vec{B} = B_0\vec{e}_z$ (\vec{e}_z is the unit vector along the z direction), perpendicular to the plane.

- (a) (*Classical E&M*) Clearly, the scalar potential ϕ can be taken as zero, for this problem. Find the vector potential \vec{A} . There may be many ways to do this. I recommend the way like in pages 3 and 4 of LN 14, using the cylindrical symmetry of the problem.
- (b) (*Newtonian mechanics*) Briefly, but sufficiently, explain *why* the electron undergoes a *uniform circular motion* and how its orbit size, R , is determined by v (speed), and B_0 while the angular frequency ω_c is determined by B_0 only (plus other fundamental constants of nature, of course). Find R and ω_c . Also, specify the direction of the circular motion.
- (c) (*Semi-classical physics*) Use the Bohr-Sommerfeld quantization rule

$$\oint p_\theta d\theta = (n + \gamma)h \quad n = 0, 1, 2, \dots \quad (n \gg 1)$$

(p_θ is the angular momentum associated with θ . Maybe a better notation for p_θ is L_θ . In any case, $p_\theta = L_\theta = (\vec{r} \times \vec{p})_z$.) to obtain (i) the quantized energy values ($E_n = \frac{1}{2}m_e v^2$), and (ii) the quantized magnetic flux ($\int d\vec{\sigma} \cdot \vec{B}$) enclosed by the semi-classical orbit, whose increment must be the *flux quantum* $\Phi_0 = h/e = 4.14 \times 10^{-15} \text{ T}\cdot\text{m}^2$.

- (d) (*Quantum mechanics – Landau levels*) Use the vector potential that you obtained above, to solve the problem quantum mechanically (cf. Eq. 14.13). This problem is exactly solvable. Show that the problem maps to a one dimensional simple harmonic oscillator, and obtain the quantized energy in terms of the $\hbar\omega_c$. If you are doing the problem with a vector potential with the cylindrical symmetry, then examine the commutator $[\hat{p}_y + \frac{e}{2}B_0\hat{x}, \frac{1}{eB_0}(\hat{p}_x - \frac{e}{2}B_0\hat{y})]$.
- (e) (*Flux/energy quantization*) Show that for a certain value of γ , which you must identify, the semi-classical result for E_n becomes the exact quantum result. The $\langle \pi\hat{r}^2 \rangle$ part is removed, since it is quite involved. Also, a different γ value (0) is required to fit the semi-classical result to the quantum result of $\langle \pi\hat{r}^2 \rangle$.

Problem 3 (20 points; **Dipole approximation**) Prove Eq. 15.5 and then, using that result, prove Eq. 15.7.

Problem 4 (30 points; **Stimulated emission**) Derive Eqs. 15.24 and 15.25 following the instruction given right before them.

Problem 5 (30 points) Problem T9.14

Problem 6 (40 points; extra credit) Problem T9.11

Problem 7 (40 points; extra credit) Given the quantum operator Eq. 14.37 for photon, calculate the free photon Hamiltonian

$$\hat{H} = \int_{\mathcal{V}} d^3\vec{r} u(\vec{r}, t)$$

where the energy density u is given by Eq. 14.31 and \vec{E} and \vec{B} are given by Eqs. 14.29 and 14.30, but now with the quantized A_0 operator and its “complex conjugate” (i.e. Hermitian conjugate) as given by Eq. 14.37. This Hamiltonian corresponds to the energy of only one mode of photon with definite values of \vec{k} and $\vec{\varepsilon}$. Show that the result is given as

$$\hat{H} = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \hbar\omega \quad (1)$$

where the subscripts $(\vec{k}, \vec{\varepsilon})$ for \hat{a} and \hat{a}^\dagger are omitted for brevity (cf. Eq. 14.37). Note that time does not appear in this equation: i.e., you must show that the integral $\int_{\mathcal{V}} d^3\vec{r} u(\vec{r}, t)$ is time independent. Consider the volume \mathcal{V} as a cube of equal sides L ($L^3 = \mathcal{V}$), and apply the periodic boundary condition (cf. page 6 of LN 15).

(Note) The total Hamiltonian for all free photon modes is then given by

$$\hat{H} = \sum_{\vec{k}, \vec{\varepsilon}} \left(\hat{a}_{\vec{k}, \vec{\varepsilon}}^\dagger \hat{a}_{\vec{k}, \vec{\varepsilon}} + \frac{1}{2} \right) \hbar\omega_{\vec{k}}$$