Due Oct. 9, Tuesday

Problem 1 (10 points) Consider two operators $\hat{O}_1$ and $\hat{O}_2$. We say that they are equal to each other if all matrix elements for them are equal, i.e., if

$$\left\langle \alpha \left| \hat{O}_1 \right| \beta \right\rangle = \left\langle \alpha \left| \hat{O}_2 \right| \beta \right\rangle$$

for any states $|\alpha\rangle$ and $|\beta\rangle$.

(a) Given the above definition, prove the following: $\hat{O}_1$ and $\hat{O}_2$ are equal *if and only if*

$$\left\langle \alpha \left| \hat{O}_1 \right| \alpha \right\rangle = \left\langle \alpha \left| \hat{O}_2 \right| \alpha \right\rangle$$

for any state $|\alpha\rangle$. [Hint: consider $|\alpha\rangle = |\alpha_1\rangle + |\alpha_2\rangle$ and then $|\alpha\rangle = |\alpha_1\rangle + i\,|\alpha_2\rangle$ for given arbitrary states $|\alpha_1\rangle$ and $|\alpha_2\rangle$.]

(b) Suppose that an observable, $\hat{\xi}$, has a mixed eigenvalue spectrum, some discrete eigenvalues $\{\xi_i\}$ and some continuous $\{\xi\}$. Its eigenvectors are (Dirac-)normalized.

$$\begin{aligned} \langle\, \xi_i \,|\, \xi_j \,\rangle &= \delta_{i,j} \\ \langle\, \xi \,|\, \xi' \,\rangle &= \delta(\xi - \xi') \\ \langle\, \xi \,|\, \xi_i \,\rangle &= 0 \end{aligned}$$

Consider an arbitrary physical state $|\alpha\rangle$. Show, using the result of (a), that the resolution of identity

$$\sum_i |\xi_i\rangle \langle\xi_i| + \int d\xi\, |\xi\rangle \langle\xi| \;=\; 1$$

can be proved readily using the condition that the total probability of all possible outcomes of measuring $\hat{\xi}$ is unity for any physical state.

Problem 2 (15 points) Equations (2.20) through (2.25) have been derived assuming that the guitar string is infinite in length ($-\infty$ to ∞). However, for a real guitar string, we must restrict $x = 0$ to L, and also apply the boundary condition that any wave of the string must vanish at $x = 0$ and L. Accordingly, values of allowed k values are discrete.

(a) Find these discrete k values, equations for these discrete ~~plane wave~~ k-basis (sine wave), corresponding to equations (2.20) through (2.25).

(b) What would be a QM problem that is exactly equivalent in math to this guitar string problem?

Problem 3 (10 points) Fill in the steps for proving the conservation principles stated in the box of page 20 of LN 2, starting from the two equations given immediately prior to the box.

Problem 4 (15 points) Consider an electron in three dimensions. Its potential energy $V(\vec{x})$ is given in each case below. In each case, (1) indicate which of the following nine symmetries exist for the Hamiltonian of the system: translational symmetries along three axes, rotational symmetries along three axes, reflection symmetries (parity symmetries) along three axes, (2) group symmetry operations into compatible ones and (3) find the general form of the energy eigenstate for each group of compatible symmetries.

(a) $V(\vec{x}) = 0$ (free particle)

(b) $V(\vec{x}) = V(r)$ (central potential)

(c) $V(\vec{x}) = V(z)$

Problem 5 (15 points) It is known that a spin 1/2 wave ~~function~~ state $|\chi\rangle$, if rotated 360 degrees, becomes $-|\chi\rangle$.

(a) Prove this fact, in as simple a way as you can, starting from our knowledge of the spin rotation operator (Section 2.1.4).

(b) Find the explicit matrix (in the usual $\hat{S}_z$ representation) for the spin rotation operator for rotating a half spin by an arbitrary amount θ around the z axis.

Problem 6 (20 points) Consider a spin 1/2 particle. At time $t = 0$, we measure $\hat{S}_y$ and find a value of $\hbar/2$ for its eigenvalue. Immediately after this measurement, we apply a uniform time *dependent* magnetic field parallel to the z-axis. The B-field is chosen such that the Hamiltonian is:

$$\hat{H}(t) = A(t)\hat{S}_z$$

where

$$A(t) \;=\; \omega_0 t/T \quad \text{for } 0 \le t \le T, \text{ and } \quad 0 \quad \text{for } t > T.$$

(a) Set up and solve the (time-dependent) Schrödinger equation for the spin-1/2 wave function. Show that at time t, the wave function is:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{i\theta(t)}\,|\alpha\rangle + ie^{-i\theta(t)}\,|\beta\rangle\right]$$

where $|\alpha\rangle$ and $|\beta\rangle$ are eigenstates of $\hat{S}_z$ with eigenvalues $\pm\hbar/2$, respectively, and $\theta(t)$ is a real function of time that you should determine explicitly.

(b) At a time $t > T$, we measure $\hat{S}_y$ again. What are the possible results of this measurement and with what probabilities? Find a relation between ω_0 and T such that the measurement of $\hat{S}_y$ yields a unique result.

For the last two problems, Pauli matrices may be helpful: $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.