How to take this exam

1. **This exam must be taken alone.** You should not consult with anyone about the exam, through any on-line or off-line communication, with one exception – you can ask clarifying questions or procedural questions to me, by email, by posting question on the forum, in person, or by phone. However, you can learn from any static, or quasi-static, materials on-line or off-line – but your solution must explain whatever you may have learned basing off of our course materials (see next).
2. **Show sufficient minimal derivation or explanation for all your answers.** You can refer to equations in lecture notes or the textbook. However, anything going beyond them must be derived.
3. **This exam is due 5 PM, December 13, 2012, Thursday.** Your solutions must be handed in to me personally. Please do *not* leave them in my mailbox. I will be in my office from 9 AM to 5 PM on December 13, and for the most part on December 11. However, I will not be in my office on December 12, unless a special arrangement can be made on an as-needed basis.
4. **I am available to answer any questions that you may have about the exam.** Those questions that try to clarify exam questions will be answered. The same applies to those questions that ask about procedural aspects of the exam. Questions asking about how to do problems will not likely be answered. However, for certain problems, it may be that some additional hint will be offered based on questions received.
5. **I am available by email, on the forum, in my office (but not on Dec 12 – see above), or by phone.** For an urgent question, you can use my cell phone number: 510-685-3979.
6. **Good organization of your solutions will be very important for this exam.** Please be very conscientious about it. Consider this exam as your final report, in which you try to show your best of the best. If your solutions are not easily readable, you may not earn points commensurate with your knowledge. Both neat writing and logical order are required.
7. **This exam consists of 5 problems, all required.**

Good luck!

Problem 1 (50 points) Write down the Fermi's golden rule expression for the transition rate when a sinusoidally time-dependent perturbation with frequency ω is applied, including the time-independent case ($\omega = 0$). We consider both the initial state and the final state as unique. Then, discuss the physical meaning of the δ function term that involves frequencies. Your discussion must address the question why that delta function term, and indeed the Fermi's gold rule, arises only in the long time limit, and what the "long time limit" means in precise terms.

Problem 2 (100 points) Consider the following potential for a particle moving in one dimension.

$$V(x) = \begin{cases} -ml\omega^2 x & x < 0 \\ \frac{1}{2}m\omega^2 x^2 & x > 0 \end{cases}$$

where m is the mass of the particle, $l > 0$ is a length scale constant, and $\omega > 0$ is an angular frequency scale constant. Use the Bohr-Sommerfeld quantization rule to estimate quantized energy values. The quantization condition for E may not be solvable for E and, if so, can be left as an implicit equation for E.

Problem 3 (150 points) Consider $n = 2$ energy levels of the Hydrogen atom. We consider Eq. 6.12 with $Z = 1$ as the unperturbed Hamiltonian. In this problem, we will be concerned with the orbital degree of freedom, pretending that the spin does not exist, as the spin effect can be included with ease later on.

For this problem, it is important for your solution to show precise arguments regarding the degenerate or non-degenerate nature of the perturbation theory.

(a) Consider a perturbation $V(\vec{r}) = \beta z^2$, with a constant $\beta > 0$. Use the first order perturbation theory to calculate the energy shift and the relative magnitudes of the energy splitting due to this perturbation.

(b) A weak magnetic field, $\vec{B} = B_0\vec{e}_3$, is applied along the z direction, where B_0 is a constant. Find, using the first order perturbation theory on the results of the previous part, the energy shifts due to the magnetic field.

Problem 4 (200 points) Consider a Hamiltonian for an electron in two dimensions

$$\hat{H} = \frac{2v_0}{\hbar}\hat{\vec{p}}\cdot\hat{\vec{S}} = \frac{2v_0}{\hbar}\left(\hat{p}_x\hat{S}_x + \hat{p}_y\hat{S}_y\right)$$

where $\hat{\vec{S}}$ is the spin 1/2 operator, $\hat{\vec{p}}$ is the momentum operator, and $v_0 > 0$ is a constant representing a velocity scale.

(a) Does this Hamiltonian conserve the linear momentum $\hat{p}_x$? How about $\hat{p}_y$? Explain.

(b) Does this Hamiltonian conserve $\hat{S}$, the magnitude of the spin angular momentum? Explain.

(c) Does this Hamiltonian conserve $\hat{S}_x$? How about $\hat{S}_y$ or $\hat{S}_z$? Explain.

(d) Find all eigenstates of $\hat{H}$, making use of what you have found in previous parts. [Hint: you probably need to solve a 2×2 matrix problem, with *complex* off-diagonal elements.]

(e) For the energy eigenstates, show that $\left\langle \hat{\vec{p}} \right\rangle$ and $\left\langle \hat{\vec{S}} \right\rangle$ are either parallel or anti-parallel to each other, by *explicitly analyzing* the nature of the wave function that you have obtained in the previous part[1].

(f) [Extra credit; 50 points] This type of Hamiltonian occurs in a real material, and the backward elastic scattering from $\vec{k}$ to $-\vec{k}$ is mainly responsible for the temperature dependence of the resistivity. Explain, then, why the Berry's phase argument (cf. HW 8.6) implies that the temperature dependence of the resistivity is zero if we make a simple assumption that the backward elastic scattering is totally responsible for the temperature dependence of the resistivity and the backward scattering processes involving different ways of rotating the spin have the exactly the same[2] quantum amplitude except for the Berry's phase.

Problem 5 (200 points) Consider a radial delta function potential in three dimensions

$$V(r) = \frac{\hbar^2 \gamma}{2m} \delta(r - R) \qquad\qquad \gamma > 0,\ R > 0$$

We shall consider s *orbitals only.* The total Hamiltonian is given by the above potential energy plus the kinetic energy $p^2/(2m)$. Note that the knowledge of the spherical Bessel functions is *not* a requirement at all for this problem, since we consider s orbitals only.

(a) Find all s-wave eigenstates by writing down the solutions for the radial equation in the two regions $0 < r < R$ and $r > R$, and then using the boundary condition at $r = 0$ and the (dis-)continuity of the wave function and its derivative at $r = R$. [Hint: It will reduce the complexity of equations to solve if you ignore the overall normalization factor, whose exact value is not necessary to know and so can be left unspecified for the purpose of this problem. I.e., you can put one of the coefficients as unity, and determine other coefficients by boundary condition and the (dis-)continuity.]

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[1] An argument based on the eigenvalues alone is also possible. But, here, you are *required* to provide your arguments based on the properties of the *eigenstates* alone. In reality, this would be a way to double-check that your eigenstates are correct.

[2] This would be the case if the interaction that causes the scattering is a spin-independent interaction.

(b) If $\gamma \to \infty$, the radial wave vector (k) value and the energy eigenvalue (E) become quantized. Find those quantized values of k and E.

(c) Let us assume $\gamma \gg k$, while kR can be any finite value. Find the phase shift δ_0 from your solution to part (a). (i) Show that δ_0 behaves just like that of the hard sphere problem ($\delta_0 \approx -kR$) if kR is is not near any quantized value found in part (b). (ii) On the other hand, if kR is near a quantized value found in part (b), then show that the following **resonance scattering** behavior is obtained.

1. The cross section has a *Lorentzian profile*[3]:

$$\sigma \approx \left(\frac{2\pi}{k}\right)^2 \frac{1}{\pi} \frac{\frac{k^4}{\gamma^4}}{\left(\tilde{k}R + \frac{k}{\gamma}\right)^2 + \frac{k^4}{\gamma^4}}$$

   for small $\tilde{k}R$, where $\tilde{k} \equiv k - k_Q$ and k_Q is any quantized value from part (b). Note that the resonance is very sharp for large γ.

2. The phase shift is given by $\frac{\pi}{2}$, on resonance, corresponding to the maximum σ.

Note that any phase shift is defined only modulo π.

(d) [Extra credit; 50 points] Show that $\delta_0 = \pi/2$, for which the cross section becomes the maximum, can occur even when kR is not near any quantized value of part (b). Show however that in that case there is no resonant profile like the above. Show that for this non-resonant case $\delta_0 = \pi/2$ is reached *from above*, while for the resonant case ($k \approx k_Q$), $\delta_0 = \pi/2$ is reached *from below*, and goes very rapidly from 0 to $\pi/2$ to π, as kR passes through, while *increasing*, the value corresponding to $\delta_0 = \pi/2$.

This type of resonance behavior is obtained whenever a scattering state forms a **virtual bound state**, i.e., a state that will become a true bound state should the tunneling between the bound state at small r and the extended state at large r is made to vanish.

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[3] The Lorentzian line shape is typical of any resonance problem, not just this problem.