Due 10 AM, June 8, Monday
Perfect score = 40 points; Extra credit = 40 points; Maximum score = 80 points.

Problem 1 (10 points) (Lorentz transformation) The Lorentz transformation is usually written as, with $\gamma = (1 - (V^2/c^2))^{-1/2}$,

$$x' = \gamma(x - Vt),$$
$$t' = \gamma\left(t - \frac{Vx}{c^2}\right),$$

with $y' = y$ and $z' = z$. These formulas can be generalized to the case when the relative velocity is not necessarily parallel to the x (or x') axis. Find the Lorentz transformation in terms of $t, \vec{r}$, $t', \vec{r}'$, and $\vec{V}$. That is, your answer should be expressed in a spatial-component-free way, and it should consist of two equations that show how t and $\vec{r}$ are converted to t and $\vec{r}'$ through γ and the inner product (and possibly the cross product) between $\vec{r}$ and $\vec{V}$.

Problem 2 (20 points) (Vector and Pseudo-vector) Vectors are objects that transform like the coordinate vector (or the gradient vector) for all norm-conserving transformations (norm = length; the square-root of the "interval" in spacetime is also an example of norm). We shall consider vectors in *the ordinary 3-space* in this problem. This is the space defined by coordinate vectors $\vec{r} = (x, y, z)$. The norm-conserving transformations in this space are rotations and three reflections ($x \to -x$, $y \to -y$, or $z \to -z$; note that the inversion operation can be accomplished by successive reflections and need not be included here; however you may note, and make use of, the fact that rotations and the inversion can generate all rotations and reflections in the case of this 3-space). They are represented by orthogonal matrices.

(a) Show that in this space the distinction between contra-variant vector (vector that transforms like a coordinate vector) and co-variant vector (vector that transforms like a gradient vector) is unnecessary, since $\vec{r}$ and $\frac{\partial}{\partial \vec{r}}$ transform the same way. [Notes: (1) You are asked to show that if $\vec{r}' = \overleftrightarrow{O}\vec{r}$, then $\frac{\partial}{\partial \vec{r}'} = \overleftrightarrow{O}\frac{\partial}{\partial \vec{r}}$, also, for any $\vec{r}$-independent orthogonal matrix $\overleftrightarrow{O}$, and (2) the metric tensor of this space is a 3×3 unit matrix, which is, of course, consistent with the fact that there is no distinction between co-variant and contra-variant vectors.]

(b) However, even in this space, there are vectors and then there are pseudo-vectors. Pseudo-vectors are objects that act like vectors for all orthogonal transformations except reflections, for each of which they act in the opposite manner (e.g., when the sign of the x coordinate is changed, the y and z components of a pseudo-vector are changed in sign). Show that the momentum $\vec{p} = m\gamma\vec{v}$ is a vector, the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is not a vector, but a pseudo-vector. [Hint: to prove that $\vec{L} = \vec{r} \times \vec{p}$ does transform as a vector for rotations, this expression for the vector product $L_i = \epsilon_{ijk} x_j p_k$ may be useful, where the repeated indices are summed over (the Einstein summation notation), $j, k = 1, 2, 3$ (Roman subscripts = 1, 2, 3 for 3-space; Greek subscripts/superscripts = 0, 1, 2, 3 for 4-spacetime).]

(c) From Newton's law, prove that the force $\vec{F}$ is a vector. Using the Lorentz force law, show that this means $\vec{E}$ is a vector but $\vec{B}$ is a pseudo-vector.

(d) Now, consider a constant rank-2 tensor in this space. Show that any such tensor can be written as a constant number times δ_{ij}. [Note: there is no constant non-zero rank-1 tensor, i.e., you cannot find a non-zero vector that will not change its value on any rotation or reflection.]

(e) Consider the Levi-Civita symbol, ϵ_{ijk}, which, *by definition*, takes the same values before and after rotation or reflection. Show that, as defined thus, ϵ_{ijk} is a rank-3 *pseudo*-tensor. That is, show that, when you in fact rotate the coordinate system (or vectors), then ϵ_{ijk} remains invariant, but it flips sign on a single reflection. [Hint: think in terms of the determinant of a matrix.][Note: by the previous part, there is no such thing as a constant non-zero rank-3 tensor.]

(f) Show that $\epsilon_{ijm}\epsilon_{klm}$ is a rank-4 tensor and is equal to $\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. [Note: the fact that this constant rank-4 tensor is written as a sum of products of δ_{ij}'s stems from the results of part (d).]

Problem 3 (20 points) (Doppler effect of light) Suppose that you are detecting light emitted from a moving radiation source. You measure the wave vector $\vec{k}$ and the angular frequency ω. The angular frequency that would be measured in the rest frame of the radiation source is ω_0. The Doppler effect of light is given by

$$\omega = \frac{\omega_0}{\gamma \left(1 - \dfrac{\vec{v} \cdot \hat{k}}{c} \right)}$$

where $\vec{v}$ is the relative velocity of the source of radiation to your reference frame, $\hat{k}$ is the unit vector along $\vec{k}$, and $\gamma = 1/\sqrt{1 - (v^2/c^2)}$. The derivation of this formula can be given in many different ways (two of which were given during class—the period point of view and the $(\omega, \vec{k})$ 4-vector point of view). Here in this problem, you are asked to derive the above formula from the "λ point of view" as follows. A conceptual way to measure λ would be to take a snapshot of the wave taken at a fixed time and measure the periodicity of the wave pattern. Use this method to derive the above formula. Do *not* assume that $\vec{v}$ and $\vec{k}$ are co-linear (you may need to use the result of problem 1).

Problem 4 (10 points) (Muon decay) Problem 12.7.

Problem 5 (20 points) (Liénard formula) Problem 12.71.