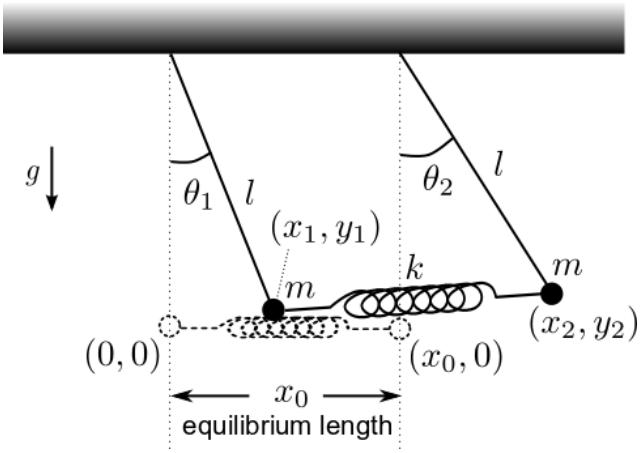
Notes for Lecture 17

Coupled linear oscillators, cont.

We have formulated the coupled linear oscillator problem completely in the last lecture. Here in this note, we explore some examples. As we do, we will notice that some normal mode frequencies vanish. These zero frequency normal modes do not represent vibration, but they represent completely different types of motion such as translation or rotation.

17.1 Coupled pendulum



Here, we consider two identical masses that are connected by a spring of equilibrium length x_0. Each mass is connected to a massless rigid rod of length l and forms a simple pendulum, which can swing in the xy plane as shown in the above

digram. In equilibrium, the two simple pendulums are vertical, and the two masses are separated horizontally by distance x_0, as shown.

We will assume small angle motion: $|\theta_1| \ll 1$ and $|\theta_2| \ll 1$.

Let us solve this problem for normal modes.

As indicated in the above diagram, the origin of the xy plane shown is taken to the equilibrium position of the mass on the left. Then, we get the following coordinates

$$(x_1, y_1) = (l\sin\theta_1, l(1-\cos\theta_1)), \tag{17.1}$$
$$(x_2, y_2) = (x_0 + l\sin\theta_2, l(1-\cos\theta_2)). \tag{17.2}$$

From this, the kinetic energy of this problem is easily calculated (and the result is as expected)

$$K = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2. \tag{17.3}$$

To calculate the potential energy due to the spring, we must calculate the distance between the two masses first. Let us call that distance d.

$$d = \sqrt{(x_0 + l(\sin\theta_2 - \sin\theta_1))^2 + l^2(\cos\theta_1 - \cos\theta_2)^2}, \tag{17.4}$$
$$= \sqrt{x_0^2 + 2lx_0(\sin\theta_2 - \sin\theta_1) + l^2(\sin\theta_1 - \sin\theta_2)^2 + l^2(\cos\theta_1 - \cos\theta_2)^2}. \tag{17.5}$$

For small angle, we take only the leading order term in θ_1 and θ_2. In the square root, the linear terms come from the second term, and so the third and fourth terms can be ignored completely.

$$d \approx \sqrt{x_0^2 + 2lx_0(\theta_2 - \theta_1)} \tag{17.6}$$
$$= x_0 + l(\theta_2 - \theta_1). \tag{17.7}$$

Since the potential energy is given by $\frac{1}{2}k(d-x_0)^2$, we get

$$U_{spring} \approx \frac{1}{2}kl^2(\theta_2 - \theta_1)^2. \tag{17.8}$$

To this, we must add the gravitational potential energy, which is given by

$$U_{gravity} = mg(y_1 + y_2) \tag{17.9}$$
$$= mgl(1 - \cos\theta_1 + 1 - \cos\theta_2) \tag{17.10}$$
$$\approx \frac{1}{2}mgl(\theta_1^2 + \theta_2^2). \tag{17.11}$$

By adding up these two potential energies, we get

$$U = \frac{1}{2}kl^2(\theta_2 - \theta_1)^2 + \frac{1}{2}mgl(\theta_1^2 + \theta_2^2). \tag{17.12}$$

So, the stiffness tensor is given by

$$\overleftrightarrow{A} = \begin{pmatrix} mgl + kl^2 & -kl^2 \\ -kl^2 & mgl + kl^2 \end{pmatrix}, \tag{17.13}$$

while the mass tensor is given by

$$\overleftrightarrow{M} = \begin{pmatrix} l^2 m & 0 \\ 0 & l^2 m \end{pmatrix} = l^2 m \overleftrightarrow{1}, \tag{17.14}$$

where $\overleftrightarrow{1}$ is the unit matrix. Notice that the mass tensor is very simple, and so the eigenvalue equation

$$\overleftrightarrow{A}\vec{T} = \lambda \overleftrightarrow{M}\vec{T}, \qquad \lambda \equiv \omega^2 \tag{17.15}$$

can be rewritten as

$$\frac{1}{ml^2}\overleftrightarrow{A}\vec{T} \equiv \overleftrightarrow{a}\vec{T} = \lambda \vec{T}. \tag{17.16}$$

Note that we have removed the subscript index i from $\vec{T}$ and λ (ω^2) for simplicity; we know that we must get two eigenvalues and corresponding two eigenvectors. Here, the matrix $\overleftrightarrow{a}$ is given by

$$\overleftrightarrow{a} = \begin{pmatrix} \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l} + \frac{k}{m} \end{pmatrix} = \begin{pmatrix} \omega_g^2 + \omega_k^2 & -\omega_k^2 \\ -\omega_k^2 & \omega_g^2 + \omega_k^2 \end{pmatrix}. \tag{17.17}$$

The secular equation $|\overleftrightarrow{a} - \lambda\overleftrightarrow{1}| = 0$ can be solved immediately to give

$$\lambda = \omega^2 = \omega_g^2 + \omega_k^2 \pm \omega_k^2 = \omega_g^2 \text{ or } \omega_g^2 + 2\omega_k^2. \tag{17.18}$$

What is the eigenvector for the soft mode ($\omega = \omega_g$)? In this case, $\overleftrightarrow{a} - \lambda\overleftrightarrow{1} = \omega_k^2 \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. Therefore, the eigenvector can be taken as (from $\overleftrightarrow{a}\vec{T} = 0$)

$$\vec{T}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \omega_1 = \omega_g = \sqrt{\frac{g}{l}}. \tag{17.19}$$

What is the eigenvector for the hard mode ($\omega^2 = \omega_g^2 + 2\omega_k^2$)? In this case, $\overleftrightarrow{a} - \lambda\overleftrightarrow{1} = -\omega_k^2 \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Therefore, the eigenvector can be taken as

$$\vec{T}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \omega_2 = \sqrt{\omega_g^2 + 2\omega_k^2} = \sqrt{\frac{g}{l} + \frac{2k}{m}}. \tag{17.20}$$

The vector $\vec{T}_i$ tells us the shape of the normal mode. For $\vec{T}_1$, we see that the θ_1 component and the θ_2 component are identical. So, this is a mode in which θ_1 and

θ_2 move in phase. For $\vec{T}_2$, θ_1 and θ_2 are opposite (1 and -1), and so this is a mode in which θ_1 and θ_2 are out of phase. To complete the solution, let us ask, what are η_1 and η_2, in terms of $_1$ and θ_2? To know this, we must invert the matrix $\overleftrightarrow{T}$, since $\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{q}$ (last lecture), where $\vec{q} = \binom{\theta_1}{\theta_2}$.

$$\overleftrightarrow{T} = \left(\vec{T}_1\vec{T}_2\right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{17.21}$$

The inverse of this matrix is

$$\overleftrightarrow{T}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{17.22}$$

To get this inverse matrix fast, one can use the fact that $\frac{1}{\sqrt{2}}\overleftrightarrow{T}$ is an orthogonal matrix, whose transpose is its inverse. Anyhow, by applying $\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{q}$, we get

$$\eta_1 = \frac{\theta_1 + \theta_2}{2} = D_1\cos(\omega_1 t + \phi_1), \tag{17.23}$$

$$\eta_2 = \frac{\theta_1 - \theta_2}{2} = D_2\cos(\omega_2 t + \phi_2). \tag{17.24}$$

Here, D_1, D_2, ϕ_1, and ϕ_2 are integration constants. And, using $\vec{q} = \overleftrightarrow{T}\eta$ (last lecture), we get

$$\theta_1 = \eta_1 + \eta_2 = D_1\cos(\omega_1 t + \phi_1) + D_2\cos(\omega_1 t + \phi_2), \tag{17.25}$$

$$\theta_2 = \eta_1 - \eta_2 = D_1\cos(\omega_1 t + \phi_1) - D_2\cos(\omega_1 t + \phi_2). \tag{17.26}$$

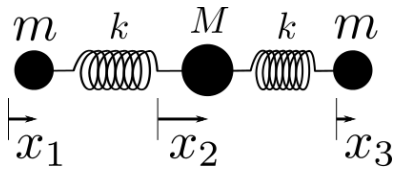
Note that the normal modes for this example have the same type of symmetry as the normal modes of the example studied in the previous lecture.

Had we normalized $\vec{T}_1$ and $\vec{T}_2$ (not required), then the matrix $\overleftrightarrow{T}$ would have turned out to be orthogonal, and we would have gotten $\eta_1 = \frac{\theta_1+\theta_2}{\sqrt{2}}$, $\eta_2 = \frac{\theta_1-\theta_2}{\sqrt{2}}$, and $\theta_1 = \frac{\eta_1+\eta_2}{\sqrt{2}}$, $\theta_2 = \frac{\eta_1-\eta_2}{\sqrt{2}}$.

17.2 Carbon dioxide

Let us pretend[1] that our classical mechanics is able to shed light on the carbon dioxide molecule. Here is a simple diagram that shows a model for such a molecule. Note that $M < m$ for the carbon dioxide molecule. Here, we leave M and m as symbols for any mass values.

[1] To be sure, classical mechanics is not appropriate for describing molecules. However, what we do here turns out to be an excellent exercise. The quantum condition can be added at the end to "quantize each normal mode" and give the completely valid quantum mechanical solution.



Clearly, this molecule problem has a total of nine degrees of freedom. Here, we shall focus on only three degrees of freedom, along the axis of the molecule, leaving the more general discussion of all degrees of freedom for a later section in this note.

The kinetic energy is given by

$$K = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}M\dot{x}_2^2. \tag{17.27}$$

The potential energy is given by

$$U = \frac{1}{2}k\left((x_1 - x_2)^2 + (x_2 - x_3)^2\right). \tag{17.28}$$

So, we get

$$\overleftrightarrow{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}, \tag{17.29}$$

$$\overleftrightarrow{A} = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{17.30}$$

Remember that $\overleftrightarrow{A}$ is made to be symmetric by design. Now the secular equation $|\overleftrightarrow{A} - \omega^2\overleftrightarrow{M}| = 0$, can be written as

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda M' & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0, \qquad \lambda \equiv \frac{m}{k}\omega^2,\ M' \equiv \frac{M}{m}. \tag{17.31}$$

This equation is rewritten as

$$(1-\lambda)\{(2 - M'\lambda)(1-\lambda) - 1\} + (-1)(1-\lambda) = (1-\lambda)\lambda\{M'\lambda - (2 + M')\} = 0.$$

Thus, we get

$$\lambda = 0,\ 1,\ \frac{2 + M'}{M'}. \tag{17.32}$$

1. $\lambda_1 = 0$ corresponds to the eigenvector $\vec{T}_1 = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, which satisfies

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0. \tag{17.33}$$

The general solution to this is $u = v = w$. By choosing $u = 1$, we get

$$\vec{T}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \omega_1 = 0. \tag{17.34}$$

In this mode, all masses are moving in step, and so there is no restoring force. This is why the frequency is zero. This mode describes the motion of the center of mass. This mode is a **translation mode**.

2. $\lambda_2 = 1$ corresponds to the eigenvector $\vec{T}_2 = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, which satisfies

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 - M' & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0. \tag{17.35}$$

The general solution to this is $u = -w$ and $v = 0$. By choosing $u = 1$, we get

$$\vec{T}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \omega_2 = \sqrt{\frac{k}{m}}. \tag{17.36}$$

Here, the central mass remains fixed, and the two outer masses are moving out of phase, creating a breathing mode. Note that the center of mass remains fixed, as expected (cf., the last section of the last lecture note).

3. $\lambda_3 = 1 + \frac{2}{M'}$ corresponds to the eigenvector $\vec{T}_3 = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, which satisfies

$$\begin{pmatrix} -\frac{2}{M'} & -1 & 0 \\ -1 & -M' & -1 \\ 0 & -1 & -\frac{2}{M'} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0. \tag{17.37}$$

The general solution to this is $u = w$ and $v = -\frac{2}{M'}u$. Again, by choosing $u = 1$, we get

$$\vec{T}_3 = \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}, \qquad \omega_3 = \sqrt{\frac{k}{m} \cdot \frac{M + 2m}{M}}. \tag{17.38}$$

In this mode, the two masses on the outside move in phase. At the same time, the mass at the center is moving out of phase. It does so that the center of mass remains fixed, again as expected.

In this example, the matrix $\overleftrightarrow{T}$ is given by collecting the above column vectors

$$\overleftrightarrow{T} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -\frac{2m}{M} \\ 1 & -1 & 1 \end{pmatrix}. \tag{17.39}$$

Note that $\vec{T}_1$ and $\vec{T}_3$ are *not* orthogonal. The reason is that the mass tensor is not proportional to a unit matrix (see the last lecture). So, we cannot make $\overleftrightarrow{T}$ an orthogonal matrix by normalizing $\vec{T}_i$'s. This means that obtaining $\overleftrightarrow{T}^{-1}$ must involve a brute force method. Here is the result.

$$\overleftrightarrow{T}^{-1} = \frac{M}{2(M+2m)} \begin{pmatrix} \frac{2m}{M} & 2 & \frac{2m}{M} \\ 1+\frac{2m}{M} & 0 & -1-\frac{2m}{M} \\ 1 & -2 & 1 \end{pmatrix} \tag{17.40}$$

From $\overleftrightarrow{T}$, we can express x_i's in terms of η_i's, by $\vec{x} = \overleftrightarrow{T}\vec{\eta}$. Then, η can be expressed in terms of x_i's, by using $\overleftrightarrow{T}^{-1}$.

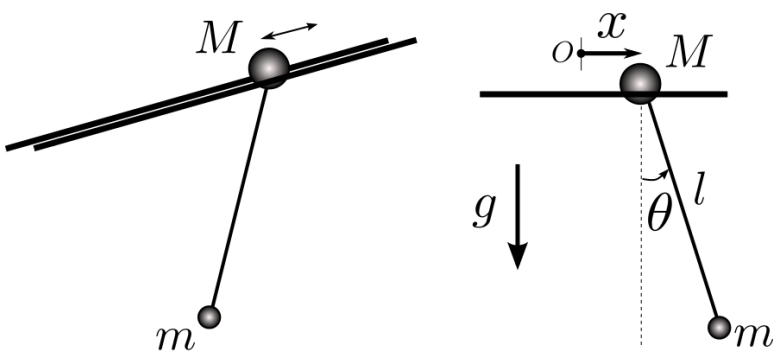
As we end this example, let us note couple of points.

1. Here, we solved a three by three matrix. We could have anticipated the translational mode solution and two internal mode solutions in the beginning. Then, for internal modes, we could have set up a two by two matrix problem, by using the constraint that the center of mass is fixed. However, the mathematical advantage gained is only marginal.

2. In general, as the matrix becomes larger, there can be degeneracies in eigenvalues. As we saw in the last lecture, we must then find correct linear combinations of degenerate eigenvectors that diagonalize the mass tensor in the degenerate subspace. However, quite often, the mass tensor is diagonal to begin with. In this case, *any* choices of degenerate eigenvectors are good, as long as they are linearly independent.

17.3 Coupled oscillator without spring

Let us imagine that a ball of mass M is sliding without friction on a horizontal rail. Underneath the ball is connected a simple pendulum consisting of massless rod and a mass m. The setup is illustrated in the following diagram.



Here, l is the distance between the centers of the two balls. This is a simple example of a coupled oscillator problem. In this case, the coupling is through the kinetic energy, not the potential energy.

In the following treatment, we shall assume that the two masses are point masses (small radii), ignoring any rotational inertia. The treatment including the rotational motion is not particularly difficult, and is left for your extra work.

We will assume small angle oscillation $|\theta| \ll 1$.

As indicated in the above diagram, the position of mass M is given by x. Now, the position of mass m requires a two dimensional coordinate, which we define as x_m and y_m. From the above diagram,

$$x_m = x + l\sin\theta, \tag{17.41}$$
$$y_m = -l\cos\theta, \tag{17.42}$$

if we take the origin of the coordinate system as point O The potential energy is simply to writing down

$$U = -mgl\cos\theta \approx \frac{1}{2}mgl\theta^2 + \text{const.} \tag{17.43}$$

We shall ignore the constant from now on. For calculating the kinetic energy, we note that

$$\dot{x}_m = \dot{x} + l\dot{\theta}\cos\theta, \tag{17.44}$$
$$\dot{y}_m = l\dot{\theta}\sin\theta, \tag{17.45}$$
$$\dot{x}_m^2 + \dot{y}_m^2 = \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2. \tag{17.46}$$

For small angles, we note that $\dot{\theta}$ is already small, and so in the second term, $\cos\theta \approx 1$ is sufficient to get the expression correct up to second order, which is what we need

in the Lagrangian. Therefore, the kinetic energy is given by

$$K \approx \frac{1}{2}\left[(M+m)\dot{x}^2 + 2ml\dot{x}\dot{\theta} + ml^2\dot{\theta}^2\right], \tag{17.47}$$

where the M term comes from the kinetic energy of mass M. Therefore, we get

$$\overleftrightarrow{M} = \begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}, \tag{17.48}$$

$$\overleftrightarrow{A} = \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}. \tag{17.49}$$

$$\tag{17.50}$$

The eigenvalue equation is given by

$$\begin{vmatrix} -\lambda(M+m) & -\lambda ml \\ -\lambda ml & mgl - \lambda ml^2 \end{vmatrix} = 0, \qquad \lambda \equiv \omega^2. \tag{17.51}$$

This gives

$$\lambda\left[Ml^2\lambda - (M+m)gl\right] = 0. \tag{17.52}$$

So, we get $\lambda = 0$ or $\lambda = \frac{M+m}{M}\frac{g}{l}$.

1. $\lambda_1 = 0$ corresponds to pure translation. The eigenvector equation for $\vec{T}_1 = \binom{u}{v}$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{17.53}$$

   The general solution is $v = 0$. So, by choosing $u = 1$, we get

$$\vec{T}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \omega_1 = 0. \tag{17.54}$$

   So, x is finite, but $\theta \equiv 0$ in this mode. This describes pure translation without any pendulum motion. So, there is no restoring force and the frequency is zero.

2. $\lambda_2 = \frac{M+m}{M}\frac{g}{l}$ means the following eigenvector equation

$$\begin{pmatrix} M+m & ml \\ ml & ml^2 - \frac{mgl}{\lambda_2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{17.55}$$

The general solution is $(M+m)u = -mlv$, which is the condition for the center of mass being stationary, since u is x component and v is θ component. By choosing $u = ml$, we get

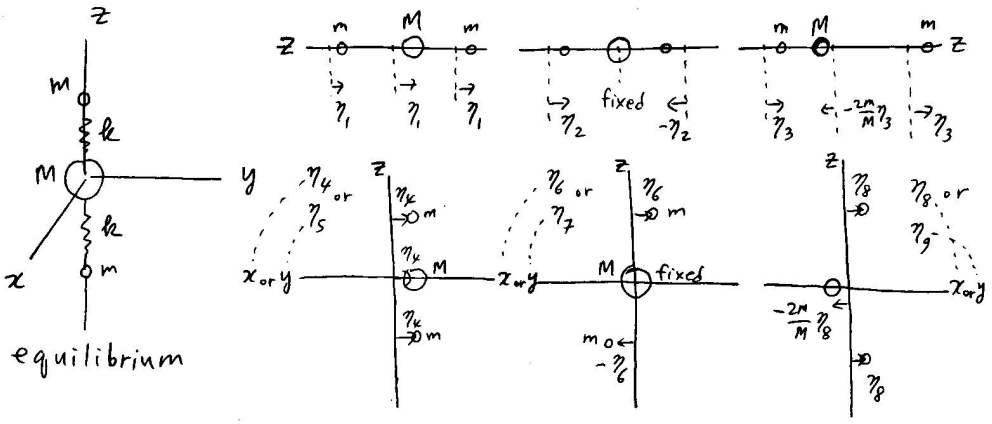
$$\vec{T}_2 = \begin{pmatrix} ml \\ -(M+m) \end{pmatrix}, \qquad \omega_2 = \sqrt{\frac{M+m}{M} \cdot \frac{g}{l}}. \tag{17.56}$$

In this mode, the motion of M and m are out of phase. If $M \to \infty$, then we see that we recover the simple pendulum frequency, as we should.

In this example also, the two vectors $\vec{T}_1$ and $\vec{T}_2$ are not orthogonal, since the mass tensor is rather complicated.

17.4 Normal modes of a molecule

Above, we studied a classical mechanics model[2] for CO_2. By choice, we only considered one dimensional motions in which any atom moves only along the line of the molecule. We obtained one translational mode (zero eigen-frequency) and two vibrational modes (finite eigen-frequencies). Of course, we should expect other modes, as each atom can move in all directions, not just along the line of the molecule. Here is a sketch of all modes.



[2] The correct quantum mechanical treatment of these molecules can be made by "simply" (what I mean is that it is simple once your learn quantum mechanics) quantizing each normal coordinate η_i after, as we discussed in the last lecture, conveniently splitting the Lagrangian, and thus the Hamiltonian, into a sum of contributions from normal modes. So, the exercise of classical coupled oscillator model for molecules is a very rewarding one, as it lays a solid foundation for the correct quantum mechanical treatment.

In the above diagram, some modes $(\eta_1, \ldots, \eta_3)$ are those we already considered. As with the η_1 mode, some modes turn out to be zero frequency oscillation—which means that it is not really an oscillation—but that is fine. Let us analyze the whole situation here, by asking and answering some questions.

1. Why do we have nine modes in total? Ans: because the total degrees of freedom $= 3 \times 3 = 9$.

2. What modes correspond to no restoring force, i.e. zero eigen-frequency? Ans: $\eta_1, \eta_4, \eta_5, \eta_6, \eta_7$.

3. What is the nature of each zero frequency mode? Ans: translation (η_1, η_4, η_5) and rotation (η_6, η_7).

4. Which modes are degenerate (i.e. of the same eigen-frequency)? Ans: All zero frequency modes are degenerate. The pair η_6, η_7 are degenerate, as are the pair η_8, η_9.

5. Which modes have a finite total momentum? Ans: translational modes (η_1, η_4, η_5) only.

6. Which modes have a finite total angular momentum? Ans: All rotational modes (η_6, η_7) and all translational modes (η_1, η_4, η_5). The latter, only if the angular momentum is measured around a point that is off the path of translation.

7. Why is it that η_6, η_7 modes have no restoring force but η_8, η_9 modes have restoring force (and thus a finite frequency)? Ans: This is an excellent question. Notice that in terms of the spring force argument all these modes $(\eta_6, \ldots, \eta_9)$ should have zero restoring force, since the length change of the spring is 2nd order, not 1st order, in η. So, there is more physics than just Hooke's laws involved with the two springs here. Microscopically, the origin can be understood as a generation of the transverse dipole moment for the η_8, η_9 modes, while no such dipole moment develops for the η_6, η_7 modes. This is the reason why there is a finite restoring force for the η_8, η_9 modes, but zero restoring force for the η_6, η_7[3]. For this reason, the potential energy for the transverse displacements x_1, x_2 and x_3 must be written as $U \propto \big[(x_1 - x_2) + (x_3 - x_2)\big]^2$.

Having done this exercise, it may not be too difficult to understand the following table, which is valid for a molecule consisting of any number of atoms (tot-DOF = total degrees of freedom).

[3]However, all this discussion is in the spirit of the *Harmonic approximation*. See later discussions in this note.

shape of molecule	Number of masses (atoms)	tot-DOF	translational modes	rotational modes	vibrational modes
line	n	$3n$	3	2	$3n-3-2$
non-line	n	$3n$	3	3	$3n-3-3$

The only difference for the two types (line or non-line) of molecules is the number of vibrational and rotational modes. It is meaningless to talk of a rotation of a linear molecule around its own axis—this is why a linear molecule has only two distinct rotations (η_6, η_7 of the above example), while a non-linear molecule has rotation modes in all three directions.

Also, note that for a linear molecule (e.g.CO_2, H_2), one can distinguish between longitudinal modes (for which atoms move along the line of the molecule) and transverse modes (for which atoms move perpendicular to the line of the molecule). For a linear molecule with n atoms, there are $n-1$ longitudinal modes (the total of n degrees of freedom within that one dimensional space minus 1 for translation). Thus, the total number of transverse vibrational modes is $3n-3-2-(n-1) = 2n-4$. It is no coincidence that this is an even number. It is for the following reason. Suppose there is a transverse mode where the displacements of all atoms are in the x direction, while the linear molecule is aligned along the z axis. Then this transverse mode has a twin mode where the displacements of all atoms, while identical in value, are in the y direction. These two modes must both exist, and they have the same eigen-frequency, i.e. they are degenerate. For either direction (x or y) of displacement, then, there will be $n-2$ normal modes. Indeed, in the above example of CO_2, there was only one transverse mode per direction.

Knowing normal modes before solving matrices

If all normal modes are obvious from the get-go, then in principle the following is all one needs to find all normal modes. Write down all normal modes (namely all $\vec{T}_i$ vectors, as pictorially given in the above diagram), and then prove, for each normal mode, that *each* non-stationary mass vibrates at the same frequency. For the six diagrams shown above, it is rather easy to do this! [Please try this!] For instance, for the 3rd digram, one can show easily that *each* of all three masses should vibrate at the angular frequency $\frac{(1+2m/M)k}{m}$, proving that this is indeed a normal mode with that angular frequency.
So, here is a question of great physical and practical importance. How can one know as much as possible about all these normal modes even before beginning to write down $\overleftrightarrow{A}$ and $\overleftrightarrow{M}$? A few guides are useful.

1. Distinguish between modes that have a finite total momentum (i.e. the center of mass is moving) and those that don't (i.e. the center of mass is fixed). The modes that do are translational modes.

2. Then, among the modes that have zero momentum, distinguish between modes that have a finite angular momentum and those that don't. Those that do are rotational modes.

3. Use "symmetry." If two masses are identical, then they can be swapped without changing physics. Then, the displacements of those two masses (say, x_1 and x_2) will appear as either the same ($x_1 = x_2$) or the opposite ($x_1 = -x_2$) in normal modes. Also, the "symmetry" is the reason why the longitudinal modes and the transverse modes can be considered separately from the get-go.

17.5 Normal modes, general remarks

In the last lecture and this one, we considered a coupled oscillator problem assuming small oscillations. This elementary treatment generally goes by the name of "**harmonic approximation**." The "stars" of this treatment were definitely those things called **normal modes**. In the harmonic approximation of coupled oscillators, all

potential energies and all kinetic energies are taken only up to the quadratic terms of generalized coordinates, and the result is that the system can be described in terms of completely independent normal modes, each of which does a simple harmonic oscillation.

It should seem both satisfying and unrealistic to have such a nice picture. It is very satisfying in the sense that there is a certain beauty to this separation of the whole complex problem into completely independent and easy bits. At the same time, you might mutter to yourself that this is a bit artificial in that any small perturbation (cubic or higher terms ignored in the Lagrangian that will introduce the "non-linearity") would spoil this ideal picture of completely independent normal modes. Indeed. Like any physics models, the harmonic approximation of coupled oscillators is a bit simplistic but immensely useful *starting point* for describing real world phenomena. For any given stable object, a small deformation will give rise to a restoring force that, in the first approximation, follows Hooke's law, and so the harmonic approximation is of relevance to *any* object.

We saw that some normal modes were "trivial." Like that of translation in which all masses move by the same amount. Or, rotation. We can call these **shape-and-size-preserving normal modes**. They are translations and rotations[4]. You take a snap shot of the system for which only translational or rotational normal modes are excited, and you will find that the system looks just the same in every detail at any time, except for the overall position and the overall orientation. These translational and rotational modes involve no restoring force, and thus are characterized by zero eigen-frequencies. Then, there are **shape-or-size-changing normal modes**. These are all the rest—vibrations, we call them. Of course, vibrations do not change the average shape of the system, within the harmonic approximation. At any given time, though, a snapshot of the system will show a distorted shape of the system.

Let us consider the following two completely different questions. First, would there be such a case when vibrational normal modes can be completely ignored? Second, would there be such a case when vibrational normal modes must be treated definitely beyond the harmonic approximation? The answers are yes and yes.

The first case is the case of a "**rigid body**." In this case, the vibrational modes are deemed unimportant and are ignored, in comparison to the translational and rotational modes. This is an important topic that we will discuss below.

[4]Perhaps there is a subtle point to keep in mind here. Strictly speaking all modes that we have been considering are small "oscillations." Of course, if the oscillation frequency is zero, then it is no longer an oscillation! The mode corresponds to a small part of translation or rotation. In the case of translation, one can simply repeat small translations to get the finite translation. For the rotation though, this is not possible. No matter, the normal modes corresponding to translation and rotation represent infinitesimal motions that are equivalent translation and rotation.

The second case, while being more difficult to discuss, is also quite interesting and important nonetheless. To appreciate this second case, consider the following statement. **Within the harmonic approximation, a coupled oscillator system will always remain "brand-new"**. As any object can be thought of as a coupled oscillator system to the first approximation, then this statement means the following. **No object will ever wear out, if it can be completely modeled as a harmonic coupled oscillator system**. Why so? In this nice picture of independent simple harmonic normal modes, all that can happen to the object in a collision, or any physical process, is the increase or decrease of energy of each normal mode. So, the object can, for example, vibrate more or less but on the average its shape or size can never change. In other words, it will never lose its average form—it never wears out! Our experience is quite the opposite—things wear out especially if they are subject to violent collision processes (baseball hit by a baseball bat, tennis ball hit by a racket, a tire rubbed against the road surface, etc). So, it must be the case that non-linear terms play an essential role in normal *wear and tear* processes. Indeed, without non-linear terms in the Lagrangian, it would be impossible to explain how things can (slightly) deform after a collision or why some materials can be lost during a collision.