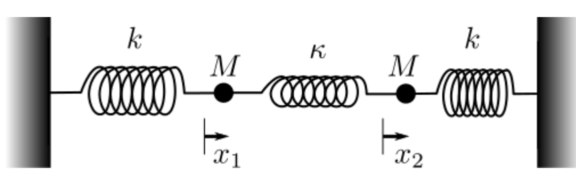
Notes for Lecture 16

Coupled linear oscillators

Now that we understand the simple harmonic oscillators and some basic physics of many particle physics, the topic of coupled linear oscillators is a good one to pick up. This is a topic of great importance.

The "linear" in the coupled linear oscillators means that we consider Hooke's law forces (which underlie simple harmonic oscillation) only and ignore any higher order forces (which are responsible for non-linear physics). This assumption is a good one, if oscillations involve only small amplitudes. So, what we are considering is coupled *small* oscillations. For the economy of language, we may often drop "small" or "linear" in the following few lectures; it should be an almost automatic assumption that in this lecture and the next lecture the assumption is that oscillations are small and the Hooke's law forces, i.e., linear forces, are the only ones to consider to a good approximation.

To begin, let us consider a simple example of two masses connected by three springs. This simple problem will act as a very good introduction.



In this example, we consider a one dimensional motion of two masses, both mass M. These two masses are connected by a spring of spring constant κ. In addition, springs of spring constant k anchor these two masses at fixed walls of some kind. We shall assume that there is no other force than spring forces.

16.1 The Newtonian method

Let us first solve this problem using the Newtonian method. It turns out that the Newtonian method can be a quicker method in many cases, and can be a much better method to use in practice. However, it may not be clear *why* the Newtonian method works. The formal general answer can be best obtained using the Lagrangian method, which we will discuss in the next section.

16.1.1 Normal modes and their frequencies

For the above problem, the equation of motion for each mass is written, by inspection,

$$M\ddot{x}_1 = -kx_1 - \kappa(x_1 - x_2), \tag{16.1}$$

$$M\ddot{x}_2 = -kx_2 - \kappa(x_2 - x_1). \tag{16.2}$$

What does "by inspection" here? In this case, it means the following. (1) The spring on the left exerts the force of magnitude $k|x_1|$. The force on mass 1 due to this spring force is $-kx_1$, the sign of which is determined by noticing the fact that the spring force opposes x_1. Now, let us consider the middle spring. The magnitude of the spring force by this spring is $\kappa|x_1 - x_2|$, since only the difference between x_1 and x_2 gives rise to a length change for this spring. The spring force exerted by this spring on mass 1 is given by $-kappa(x_1 - x_2)$, where the sign is determined by the fact that the spring force opposes the displacement x_1. For the spring force exerted by this same spring on mass 2 is given by $-\kappa(x_2 - x_1)$, since in this case[1] the force opposes x_2.

This method starts out by assuming that a solution of the following form exists.

$$x_1 = u\exp(i\omega t), \tag{16.3}$$

$$x_2 = v\exp(i\omega t). \tag{16.4}$$

This form of solution, where, all generalized coordinates oscillate at the same frequency is called a **normal mode** solution.

As in the simple harmonic oscillation problem, here we decide to do solve the problem in the complex number domain, as we expressed x_1 and x_2 in terms of complex function $\exp(i\omega t)$. Likewise, u and v are complex numbers. We have in mind that after we are done obtaining the complete solutions of x_1 and x_2, we will take their real parts, which will be the real solutions.

[1] Also note that Newton's third law is in action here.

By plugging in this normal mode solution, we get the following set of equations:

$$-M\omega^2 u = -ku - \kappa(u - v), \tag{16.5}$$

$$-M\omega^2 v = -kv - \kappa(v - u). \tag{16.6}$$

Note that the $\exp(i\omega t)$ factor has been divided out from these equations, since it is a common factor for each term.

Here, we have two coupled linear equations. That is, we have a matrix equation on hand. Indeed, these two equations can be rewritten as

$$\overleftrightarrow{B}\vec{V} = 0, \tag{16.7}$$

$$\overleftrightarrow{B} \equiv \begin{pmatrix} k + \kappa - M\omega^2 & -\kappa \\ -\kappa & k + \kappa - M\omega^2 \end{pmatrix}, \tag{16.8}$$

$$\vec{V} \equiv \begin{pmatrix} u \\ v \end{pmatrix}. \tag{16.9}$$

We like to solve the above matrix equation for $\vec{V}$. Clearly, there is a trivial solution $\vec{V} = 0$. In fact, if $\overleftrightarrow{B}$ is invertible, then we see that $\overleftrightarrow{B}^{-1}\overleftrightarrow{B}\vec{V} = \vec{V} = \overleftrightarrow{B}^{-1}0 = 0$.

Therefore, we conclude that if $\overleftrightarrow{B}$ is invertible then we will get only the trivial solution, corresponding to no motion at all. This solution is definitely not what we want.

For a non-trivial solution, then, $\overleftrightarrow{B}$ must be singular (that is, non-invertible), namely $|\overleftrightarrow{B}| = 0$, where $|\overleftrightarrow{B}|$ is the determinant of matrix $\overleftrightarrow{B}$. This gives the **secular equation** for ω^2:

$$(k + \kappa - M\omega^2)^2 - \kappa^2 = 0. \tag{16.10}$$

From which we get

$$\omega = \sqrt{\frac{k}{M}} \quad \text{or} \quad \sqrt{\frac{k + 2\kappa}{M}}. \tag{16.11}$$

These are so-called **normal mode frequencies**. They represent the natural frequencies for a coupled oscillator system.

So, using the assumption that normal mode exists, we found two of them. As we shall see soon, the reason that we found two normal modes is that there are two degrees of freedom in this problem. This is very general. If there are n degrees of freedom in a coupled linear oscillator problem, then there are n normal modes.

16.1.2 Shape of normal modes

Having obtained the normal mode frequencies, we can ask what each normal mode looks like. For this we need to solve for $\vec{V}$.

For $\omega = \sqrt{k/M}$, we see that $\overleftrightarrow{B} = \left(\begin{smallmatrix} \kappa & -\kappa \\ -\kappa & \kappa \end{smallmatrix}\right)$. So, $\overleftrightarrow{B}\vec{V} = 0$ with $\vec{V} = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ leads to the following condition:

$$u = v, \qquad \text{for } \omega = \sqrt{\frac{k}{M}}. \tag{16.12}$$

This means that the two masses will move completely in sync, or **in phase**. We call this a **symmetric mode**, since the displacement on the left and the displacement on the right are identical.

It is easy to see that why this mode has the frequency $\sqrt{k/M}$. As the spring at the center is never stretched or compressed, it is as though it does not exist!

For the other mode, $\omega = \sqrt{(k+2\kappa)/M}$, we see that $\overleftrightarrow{B} = \left(\begin{smallmatrix} -\kappa & -\kappa \\ -\kappa & -\kappa \end{smallmatrix}\right)$, which means that

$$u = -v, \qquad \text{for } \omega = \sqrt{\frac{k+2\kappa}{M}}. \tag{16.13}$$

This means that the two masses move completely **out of phase**. We call this an **anti-symmetric mode**, since the displacement on the left is opposite in sign to the displacement on the right, while equal in magnitude. This type of mode is also called a breathing mode, for an obvious reason.

It is also not so difficult to see *why* this mode has the frequency $\sqrt{(k+2\kappa)/M}$. In this mode, when the spring on the left is compressed or extended by x_1, then the spring at the center is extended or compressed by twice as much. The net result is an effective spring constant $k+2\kappa$ acting on the mass on the left. The same conclusion can be easily made for the mass on the right as well.

To prepare for the next discussion, let us write explicitly the amplitudes associted with each normal mode.

For the first normal mode, $x_1 = x_2 = u\exp(i\omega t)$. Let us take $u = D_1\exp(i\phi_1)$, with two real constants D_1 and ϕ_1. Finally, taking the real part, we can write

$$x_1 = x_2 = D_1\cos(\omega_1 t + \phi_1), \qquad \text{for } \omega = \sqrt{\frac{k}{M}} \quad \text{(soft mode)}. \tag{16.14}$$

For the second normal mode, $x_1 = -x_2 = u\exp(i\omega t)$. Let us take $u = D_2\exp(i\phi_2)$, with two real constants D_2 and ϕ_2. Taking the real part, we can write

$$x_1 = -x_2 = D_2\cos(\omega_2 t + \phi_2), \qquad \text{for } \omega = \sqrt{\frac{k+2\kappa}{M}} \quad \text{(hard mode).} \tag{16.15}$$

16.1.3 General solution

Let us note that the EOM that we started with has an important fact in common with the SHM EOM. The EOM is linear. All terms are of linear order in x_1, x_2 or their time derivatives. This means that the two normal mode solutions that we found above, if simpled added (or formed into a linear combination in general), will give another valid solution.

Keeping in mind that this problem involves two amplitudes, we realize that a solution of this problem is **a vector** $\binom{x_1}{x_2}$. Above, we obtained two such solutions for the two normal modes. If we just add them up (which is all that needs to be considered in this case as each solution already includes an arbitrary multiplicative constants D_1 and D_2), then we get another solution $\binom{x_1}{x_2}$, as given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} D_1\cos(\omega_1 t + \phi_1) \\ D_1\cos(\omega_1 t + \phi_1) \end{pmatrix} + \begin{pmatrix} D_2\cos(\omega_2 t + \phi_2). \\ -D_2\cos(\omega_2 t + \phi_2) \end{pmatrix} \tag{16.16}$$

Thus, we obtain the following general solutions for this problem

$$x_1 = D_1\cos(\omega_1 t + \phi_1) + D_2\cos(\omega_2 t + \phi_2), \tag{16.17}$$

$$x_2 = D_1\cos(\omega_1 t + \phi_1) - D_2\cos(\omega_2 t + \phi_2). \tag{16.18}$$

Why are these *general* solutions? They are solutions and they contain four integration constants $(D_1, D_2, \phi_1, \phi_2)$. These four constants can be fixed if sufficient number of initial conditions are given.

16.2 The Lagrangian method

The above method of solving Newtonian EOM works well. Even when the system involves damping, it works well. On the other hand, it does leave something to be desired as a *formalism*. For a general formalism, the Lagrangian method that we will discuss now is much more illuminating.

16.2.1 Mass tensor, stiffness tensor

In the Lagrangian method, we examine the kinetic energy and the potential energy

$$\begin{aligned} L &= K - U, \\ K &= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2, && (16.19) \\ U &= \frac{1}{2}kx_1^2 + \frac{1}{2}\kappa(x_1 - x_2)^2 + \frac{1}{2}kx_2^2. && (16.20) \end{aligned}$$

If we wrote down the Lagrangian EOM using this Lagrangian, we would obtain the precisely the same EOM as we solved above. That is reassuring, but not surprising. Here, let us turn K and U into more interesting forms.

$$\begin{aligned} K &= \frac{1}{2}\dot{\vec{x}}^t \overleftrightarrow{M} \dot{\vec{x}}, && (16.21) \\ U &= \frac{1}{2}\vec{x}^t \overleftrightarrow{A} \vec{x}, && (16.22) \\ \vec{x} &\equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. && (16.23) \end{aligned}$$

The matrices defined here are the **mass tensor** ($\overleftrightarrow{M}$) and the **stiffness tensor** ($\overleftrightarrow{A}$). Comparing the two forms of K, we can identify $\overleftrightarrow{M}$.

$$\overleftrightarrow{M} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}. \qquad (16.24)$$

For $\overleftrightarrow{A}$, let us note that

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}\kappa(x_1^2 - 2x_1x_2 + x_2^2) + \frac{1}{2}kx_2^2. \qquad (16.25)$$

For constructing $\overleftrightarrow{A}$, we take its non-diagonal terms to be *symmetric* to find

$$\overleftrightarrow{A} = \begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix}. \qquad (16.26)$$

It is very important to note that (1) by construction $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$ are real symmetric matrices and (2) $\overleftrightarrow{M}$ is a positive definite matrix. The latter means that $\vec{v}^t \overleftrightarrow{M} \vec{v}$ is positive for any non-zero vector $\vec{v}$ and zero only if $\vec{v} = 0$.

16.2.2 General formalism

In the above example, the vector $\binom{x_1}{x_2}$ is what we are solving for. In general, we can replace x's with generalized coordinates q's. In this general notation, what we are solving for is a vector of generalized coordinates

$$\vec{q} \equiv \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \tag{16.27}$$

where we assumed that the total degrees of freedom is n. A general coupled linear oscillator problem is then defined by the Lagrangian

$$L = K - U,$$
$$K = \frac{1}{2}\dot{\vec{q}}^t \overleftrightarrow{M} \dot{\vec{q}}, \tag{16.28}$$
$$U = \frac{1}{2}\vec{q}^t \overleftrightarrow{A} \vec{q}, \tag{16.29}$$

where $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$ are real symmetric matrices and, in addition, $\overleftrightarrow{M}$ is a positive definite matrix.

Generally, any system of particles in a stable equilibrium state is expected to show this form of Lagrangian near the equilibrium. This is because expanding around the equilibrium point, taking $\vec{q}$ as relative to the equilibrium, there will be no term linear in $\vec{q}$ in U. Likewise, the kinetic energy term will be at least quadratic in generalized coordinates, as it arises from v^2. If the generalized coordinates are defined such that there are higher order terms in K, then, those terms can be ignored using small angle/displacement approximation.

As one can see, the above problem covers a lot of physical situations! The description "any system of particles in a stable equilibrium state" applies to a lot of situations. In particular, molecules and solids are prime examples of this problem. This generality makes this problem very important.

16.2.3 General solution

The general approach to solve the above problem is the following. Let us consider a non-singular (i.e., invertible) coordinate transformation $\overleftrightarrow{T}$

$$\vec{q} = \overleftrightarrow{T}\vec{\eta}, \tag{16.30}$$
$$\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{q}. \tag{16.31}$$

Upon the coordinate transformation, we get

$$K = \frac{1}{2}\dot{\vec{\eta}}^t \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} \dot{\vec{\eta}}, \tag{16.32}$$

$$U = \frac{1}{2}\vec{\eta}^t \overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T} \vec{\eta}. \tag{16.33}$$

What would be really nice is if the transformed mass tensor ($\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T}$) and the transformed stiffness tensor ($\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T}$) are both diagonal. Can this wish come true? We will now show that the answer is Yes!

In order to find the necessary condition for diagonalizing both $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$, let us assume that we have found the answers in the following form.

$$\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T} = \overleftrightarrow{M}_d, \tag{16.34}$$

$$\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{A}_d. \tag{16.35}$$

Here, $\overleftrightarrow{T}$ is a non-singular matrix (thus consisting of linearly independent column vectors), not necessarily an orthogonal matrix[2], and

$$\overleftrightarrow{M}_d \equiv \begin{pmatrix} m_1^* & 0 & \cdots & 0 \\ 0 & m_2^* & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & m_n^* \end{pmatrix} \tag{16.36}$$

where m_i^*'s are **effective masses** and

$$\overleftrightarrow{A}_d \equiv \begin{pmatrix} k_1^* & 0 & \cdots & 0 \\ 0 & k_2^* & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & k_n^* \end{pmatrix} \tag{16.37}$$

where k_i^*'s are **effective spring constants**.

If $\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T} = \overleftrightarrow{M}_d$ is true, then $\overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T}\overleftrightarrow{M}_d^{-1} = 1$. Inserting this in front of $\overleftrightarrow{A}_d$ in the equation $\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{A}_d$, we get $\overleftrightarrow{T}^t\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{T}^t\overleftrightarrow{M}\overleftrightarrow{T}\overleftrightarrow{M}_d^{-1}\overleftrightarrow{A}_d$. Multiply $(\overleftrightarrow{T}^t)^{-1}$ from the left and we get

$$\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{M}\overleftrightarrow{T}\overleftrightarrow{\lambda}, \tag{16.38}$$

[2]These two equations mean that $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$ are simultaneously diagonalized by the coordinate transformation $\overleftrightarrow{T}$. If $\overleftrightarrow{T}$ is an orthogonal matrix, then they mean that $\overleftrightarrow{M}\overleftrightarrow{T} = \overleftrightarrow{T}\overleftrightarrow{M}_d$ and $\overleftrightarrow{A}\overleftrightarrow{T} = \overleftrightarrow{T}\overleftrightarrow{A}_d$: in this case, the column vectors of $\overleftrightarrow{T}$ are simultaneous eigenvectors of $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$. It is easy to see that $\overleftrightarrow{T}$ will be orthogonal if $\overleftrightarrow{M}$ or $\overleftrightarrow{A}$ is a constant times the identity matrix. This is often the case, but not always, making $\overleftrightarrow{T}$ non-orthogonal in general. So, in general, it *cannot* be said that the column vectors that make up $\overleftrightarrow{T}$ are simultaneous eigenvectors of $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$. The eigenvalue equation, correct in general, is shown in Eq. 16.40.

where

$$\overleftrightarrow{\lambda} = \overleftrightarrow{M}_d^{-1}\overleftrightarrow{A}_d = \begin{pmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \omega_n^2 \end{pmatrix}, \qquad \omega_i^2 = \frac{k_i^*}{m_i^*}. \tag{16.39}$$

Noting that the i-th column vector portion of the matrix $\overleftrightarrow{T}\overleftrightarrow{\lambda}$ is given by $\omega_i^2\vec{T}_i$, we can re-write this matrix equation as

$$\overleftrightarrow{A}\vec{T}_i = \omega_i^2\overleftrightarrow{M}\vec{T}_i \tag{16.40}$$

where $\vec{T}_i$ is the i-th column vector of $\overleftrightarrow{T}$. This eigenvalue equation is the so-called **generalized eigenvalue equation**.

With $\overleftrightarrow{A}, \overleftrightarrow{M}$ symmetric and $\overleftrightarrow{M}$ positive definite[3], the above eigenvalue equation can be shown to be always solvable to give a non-singular matrix $\overleftrightarrow{T}$. You can read more about this[4] well-known problem in any linear algebra book or a technical book such as "Numerical Recipes in C."

Eq. 16.40 is central to the formalism that we are developing here. What we have shown is that the column vectors $\overleftrightarrow{T}_i$ that diagonalize the mass tensor and the stiffness tensor, in the sense of Eqs. 16.34 and 16.35, must *necessarily* satisfy Eq. 16.40.

It turns out that Eq. 16.40 is a sufficient condition, also, showing that that equation is central to our problem at hand.

To prove the sufficiency, let us now assume Eq. 16.40. Then, we can derive Eqs. 16.34 and Eqs. 16.35. First, we note that $\vec{T}^t\overleftrightarrow{M}\vec{T}$ is a positive number, since

[3]You may wonder whether $\overleftrightarrow{A}$ is also positive definite for motions around a stable equilibrium. This is actually not true. This is because, $\vec{q}^t\overleftrightarrow{A}\vec{q}$ can be zero for non-zero $\vec{q}$, corresponding to a translational mode. On another note, notice that our formalism here would work even if the equilibrium is a (partially) unstable one. In such a case, some k_i^* values will be negative.

[4]Here is a short summary. A positive definite real symmetric matrix (like $\overleftrightarrow{M}$) has a "Cholesky decomposition": $\overleftrightarrow{M} = \overleftrightarrow{L}\overleftrightarrow{L}^t$, where $\overleftrightarrow{L}$ is a lower triangular matrix (i.e., a matrix whose entries above the diagonal are all zeroes). Thus, the eigenvalue equation is written as $\overleftrightarrow{A}\vec{T}_i = \omega_i^2\overleftrightarrow{L}\overleftrightarrow{L}^t\vec{T}_i$. Since $\overleftrightarrow{M}$ is a positive definite symmetric matrix, it is a non-singular matrix. Thus, $\overleftrightarrow{L}$ is a non-singular matrix. Namely, $\overleftrightarrow{L}^{-1}$ exists. So, multiplying $\overleftrightarrow{L}^{-1}$ from the left, we get $\overleftrightarrow{L}^{-1}\overleftrightarrow{A}\vec{T}_i = \omega_i^2\overleftrightarrow{L}^t\vec{T}_i$, which can be rewritten as $\overleftrightarrow{L}^{-1}\overleftrightarrow{A}(\overleftrightarrow{L}^t)^{-1}\overleftrightarrow{L}^t\vec{T}_i = \omega_i^2\overleftrightarrow{L}^t\vec{T}_i$. Define $\overleftrightarrow{B} = \overleftrightarrow{L}^{-1}\overleftrightarrow{A}(\overleftrightarrow{L}^t)^{-1}$, and we see that $\overleftrightarrow{B}$ is a real symmetric matrix. Define a new vector $\overleftrightarrow{L}^t\vec{T}_i = \vec{W}_i$. Then, we get $\overleftrightarrow{B}\vec{W}_i = \omega_i^2\vec{W}_i$, which is a simple eigenvalue equation for a real symmetric matrix $\overleftrightarrow{B}$. Any real symmetric matrix is diagonalizable by an orthonormal eigenvectors, and so we can find such eigenvectors $\vec{W}_i$'s and the corresponding eigenvalues ω_i^2's. Then, we can calculate $\vec{T}_i = (\overleftrightarrow{L}^t)^{-1}\vec{W}_i$.

$\overleftrightarrow{M}$ is a positive definite matrix. Second, let us transpose Eq. 16.40 and change the (arbitrary) column index from i to j to get $\vec{T}_j^t \overleftrightarrow{A} = \omega_j^2 \vec{T}_j^t \overleftrightarrow{M}$. Now, multiplying $\vec{T}_i$ to this equation from the right, and using Eq. 16.40, we get

$$\vec{T}_j^t \overleftrightarrow{A} \vec{T}_i = \omega_j^2 \vec{T}_j^t \overleftrightarrow{M} \vec{T}_i. \tag{16.41}$$

A similar equation is obtained by multiplying Eq. 16.40 by $\vec{T}_j^t$ from the left

$$\vec{T}_j^t \overleftrightarrow{A} \vec{T}_i = \omega_i^2 \vec{T}_j^t \overleftrightarrow{M} \vec{T}_i. \tag{16.42}$$

Subtracting these two equations (and swapping indices i and j just for the sake of convention), we get

$$(\omega_i^2 - \omega_j^2) \vec{T}_i^t \overleftrightarrow{M} \vec{T}_j = 0. \tag{16.43}$$

If ω_i^2's are all distinct, then this equation shows that $\overleftrightarrow{M}$ is diagonalized by $\vec{T}_i$, since $\vec{T}_i^t \overleftrightarrow{M} \vec{T}_j$ vanishes if $i \neq j$. So, we can write

$$\vec{T}_i^t \overleftrightarrow{M} \vec{T}_j = m_i^* \delta_{ij}, \tag{16.44}$$

which is exactly Eq. 16.34. Of course, there is no guarantee that all ω_i^2's are distinct. Let us suppose that all ω_i^2's are the same, in fact. Then, Eq. 16.43 does not determine any specific value for $\vec{T}_i^t \overleftrightarrow{M} \vec{T}_j$ at all. But, one can note the following: considering the matrix $\overleftrightarrow{C} = \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T}$, we see that it is a symmetric matrix, and so it can be diagonalized by an orthogonal matrix $\overleftrightarrow{O}$ such that $\overleftrightarrow{O}^t \overleftrightarrow{C} \overleftrightarrow{O}$ is diagonal. That is, we can find the orthogonal matrix $\overleftrightarrow{O}$ such that

$$\left(\overleftrightarrow{O}^t \overleftrightarrow{C} \overleftrightarrow{O}\right)_{ij} = \left(\overleftrightarrow{O}^t \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} \overleftrightarrow{O}\right)_{ij} = m_i^* \delta_{ij}. \tag{16.45}$$

Or, written a bit more explicitly in terms of the matrix sum we get

$$\left(\sum_k \vec{T}_k O_{ki}\right)^t \overleftrightarrow{M} \left(\sum_k \vec{T}_k O_{kj}\right) = m_i^* \delta_{ij}. \tag{16.46}$$

What this shows is that by forming linear combinations of old column vectors we have found a new basis vectors that diagonalize $\overleftrightarrow{M}$. Namely, if we define the new column vector set as

$$\vec{T}_j{}' \equiv \sum_k O_{kj} \vec{T}_k, \tag{16.47}$$

then we get

$$\left(\vec{T}_i{}'\right)^t \overleftrightarrow{M} \vec{T}_j{}' = m_i^* \delta_{ij}. \tag{16.48}$$

Since $\vec{T}_j$'s share the same eigenvalue $\omega^2 \equiv \omega_j^2$ for any j, by assumption, we see that any linear combination of $\vec{T}_j$'s satisfy Eq. 16.40:

$$\overleftrightarrow{A}\vec{T}_i{}' = \omega^2 \overleftrightarrow{M}\vec{T}_i{}'. \tag{16.49}$$

This means that by finding $\overleftrightarrow{O}$ we did not affect the original eigenvalue equation at all, and found a way to diagonalize $\overleftrightarrow{M}$.

One can now see that we can apply our method of solution even to a mixed case in which some eigenvalues are distinct and some eigenvalues are not. For those $\vec{T}_i$'s that share the same eigenvalue ω_i^2, we can find orthonormal linear combinations of them, by finding the orthogonal $\overleftrightarrow{O}$ matrix that has the dimension $r \times r$, where r is the total number of such $\vec{T}_i$'s that share the same eigenvalue ω_i^2. In this way, we can find $\vec{T}_i{}'$'s that replace $\vec{T}_i$'s. After the replacement, we shall forget about the original $\vec{T}_i$'s and call the new $\vec{T}_i{}'$'s $\vec{T}_i$'s, by dropping the prime. We can repeat this process for any degenerate set of $\vec{T}_i$'s, where by degeneracy we mean the fact that the same eigenvalue is shared by different eigenvectors. The result is that we can always find a set of linearly independent column vectors $\vec{T}_i$'s that simultaneously satisfy Eq. 16.40 and Eq. 16.44, no matter what the degeneracy is.

$$\vec{T}_i^t \overleftrightarrow{M}\vec{T}_j = m_i^* \delta_{ij}, \qquad \text{always possible regardless of degeneracy.} \tag{16.50}$$

Incidentally, this equation also shows that $\vec{T}_i$'s form an orthogonal set, if $\overleftrightarrow{M}$ is a constant times a unit matrix. In this case, by normalizing each vectors, the $\overleftrightarrow{T}$ matrix can be taken as an orthogonal matrix. This is *not* possible in general[5] for a real symmetric positive definite matrix $\overleftrightarrow{M}$. Also, note that due to the positive definite nature of $\overleftrightarrow{M}$, we get

$$m_i^* > 0, \qquad \text{for any } i. \tag{16.51}$$

Now that we have found that the same linearly independent set of column vectors $\vec{T}_i$'s satisfy Eqs. 16.40 and 16.50. It is easy to see how the same $\vec{T}_i$'s diagonalize $\overleftrightarrow{A}$ also. By combining Eq. 16.42 (after swapping symbols i and j) and Eq. 16.50, we get

$$\vec{T}_i^t \overleftrightarrow{A}\vec{T}_j = \omega_i^2 m_i^* \delta_{ij} \equiv k_i^* \delta_{ij}. \tag{16.52}$$

This is exactly Eq. 16.35 through our definition $k_i^* = m_i^* \omega_i^2$. This completes the proof of the sufficiency.

Note that since $m_i^* > 0$, $\omega_i^2 \geq 0$ if $k_i^* \geq 0$. This would be the case for motions around a stable equilibrium point. We will consider this case only in this note[6].

[5]However, note that $\vec{T}_i$'s form an orthogonal set in the metric space, whose metric is given by the metric tensor $\overleftrightarrow{M}$.

[6]But, see footnote 3.

In conclusion, our primary goal in a coupled linear oscillator problem is to find the eigenvalues and the eigenvectors for Eq. 16.40. After finding them, we might have a bit more work to do, since we have to diagonalize a sub-matrix of $\overleftrightarrow{M}$ in each degenerate sub-space if there are degeneracies. After this step, we have a set of linearly independent column vectors that diagonalize both $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$. Going back to the original problem set up by the kinetic energy and the potential energy (Eqs. 16.32 and 16.33), we can now write, by plugging Eqs. 16.44 and 16.52 into those equations,

$$K = \frac{1}{2}\sum_i m_i^* \dot{\eta}_i^2, \tag{16.53}$$

$$U = \frac{1}{2}\sum_i k_i^* \eta_i^2. \tag{16.54}$$

This means that the **normal coordinates** η_i's are completely independent dynamically, since

$$L = K - U = \sum_i L_i, \tag{16.55}$$

$$L_i = \frac{1}{2} m_i^* \dot{\eta}_i^2 - \frac{1}{2} k_i \eta_i^2. \tag{16.56}$$

Observe that each L_i is a Lagrangian for a simple harmonic oscillator. The simple harmonic motion described by each normal coordinate η_i is called the normal mode. The normal mode is described by the solution

$$\eta_i(t) = D_i \cos(\omega_i t + \phi_i), \tag{16.57}$$

where D_i and ϕ_i are two integration constants for the i-th normal mode. To find $q_i(t)$, all we need to do is to use $\vec{q} = \overleftrightarrow{T}\vec{\eta}$ (Eq. 16.30).

$$q_i(t) = \sum_i T_{ij}\eta_j(t), \tag{16.58}$$

which provides the general solution in terms of the original variables $q_i(t)$'s. If $2n$ initial conditions are given (recall that the total degrees of freedom is n), then all constants D_i and ϕ_i can be determined. Generally, initial conditions are given in terms of q_i and $\dot{q}_i$. These initial conditions can be converted to initial conditions for normal coordinates and their time derivatives, using $\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{q}$. Using the initial conditions for η_i and $\dot{\eta}_i$ thus determined, D_i and ϕ_i can be determined easily.

What we just proved should impress you. We just showed that, no matter how many particles we have and no matter how they interact with one another, the motion near the stable equilibrium is described by n independent normal modes (simple harmonic oscillation modes), where n is the total degrees of freedom. This is indeed a very impressive general result! In studies of vibrational modes of molecules and solids, this result lays down the foundational principle (to which some quantum mechanical considerations must be added).

16.2.4 Back to the example

Note that, for the simple example that we started with,

$$\overleftrightarrow{A} = \begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix} \tag{16.59}$$

and $\overleftrightarrow{M} = M\overleftrightarrow{1}$. For the generalized eigenvalue equation, Eq. 16.40, to have non-trivial solution, we must require

$$|\overleftrightarrow{A} - \omega^2 \overleftrightarrow{M}| = 0. \tag{16.60}$$

This is the so-called **secular equation**. For the current problem, it reads

$$(k + \kappa - \omega^2 M)^2 - \kappa^2 = 0, \tag{16.61}$$

which is exactly the same equation as Eq. 16.10, as required by physics. Eigenvectors can be found by going back to the eigenvalue equation, Eq. 16.40, and plugging in the eigenvalue. Let us label the two eigenvalues as

$$\omega_1 = \sqrt{\frac{k}{M}}, \tag{16.62}$$

$$\omega_2 = \sqrt{\frac{k+2\kappa}{M}}. \tag{16.63}$$

For $\omega = \omega_1$,

$$\overleftrightarrow{A} - \omega^2 \overleftrightarrow{M} = \begin{pmatrix} \kappa & -\kappa \\ -\kappa & \kappa \end{pmatrix}. \tag{16.64}$$

And, so its eigenvector, $\vec{T}_1$ is determined by

$$\kappa \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0, \qquad \vec{T}_1 \equiv \begin{pmatrix} u \\ v \end{pmatrix} \tag{16.65}$$

This has the general solution $u = v$. So, we can choose $u = 1$ and $v = 1$. For $\omega = \omega_2$,

$$\overleftrightarrow{A} - \omega^2 \overleftrightarrow{M} = \begin{pmatrix} -\kappa & -\kappa \\ -\kappa & -\kappa \end{pmatrix}. \tag{16.66}$$

and the eigenvector for this is easily obtained, also. Here is a complete set of solutions for this problem. We go back to the x notation for the coordinates, here. Please make sure that you recognize that these solutions agree perfectly with the solutions obtained by the Newtonian method in Section 16.1.

$$\vec{T}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{16.67}$$

$$\vec{T}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{16.68}$$

$$\overleftrightarrow{T} = \begin{pmatrix} \vec{T}_1 & \vec{T}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{16.69}$$

$$\vec{x} = \overleftrightarrow{T}\vec{\eta} \tag{16.70}$$

$$x_1 = \eta_1 + \eta_2 = D_1 \cos(\omega_1 t + \phi_1) + D_2 \cos(\omega_2 t + \phi_2) \tag{16.71}$$

$$x_2 = \eta_1 - \eta_2 = D_1 \cos(\omega_1 t + \phi_1) - D_2 \cos(\omega_2 t + \phi_2) \tag{16.72}$$

$$\overleftrightarrow{T}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{16.73}$$

$$\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{x} \tag{16.74}$$

$$\eta_1 = \frac{1}{2}(x_1 + x_2) = D_1 \cos(\omega_1 t + \phi_1) \tag{16.75}$$

$$\eta_2 = \frac{1}{2}(x_1 - x_2) = D_2 \cos(\omega_2 t + \phi_2) \tag{16.76}$$

As for the $\overleftrightarrow{T}$ matrix, we see that each column vector $\vec{T}_i$ is sort of equivalent to the vector $\vec{V}$ (Eq. 16.9) found in the Newtonian method section. Indeed, they are the same eigenvectors that satisfy the same eigenvalue equation. Just as the $\vec{V}$ vector in the Newtonian formalism, there is an overall scale that is freely adjustable for each eigenvector $\vec{T}_i$ in the Lagrangian formalism. **However, there is an important difference between them.** Each $\vec{T}_i$ should be a *real* vector and it should be chosen definitely as one particular non-zero vector. Namely, each $\vec{T}_i$ should be chosen as one specific vector that represents a direction in the n dimensional vector space. Often, it is a good idea to choose $\vec{T}_i$ as a unit vector, if that makes $\overleftrightarrow{T}$ an orthogonal matrix. Often, it is OK to choose $\vec{T}_i$'s to have nice numbers (all integers). It is your choice.

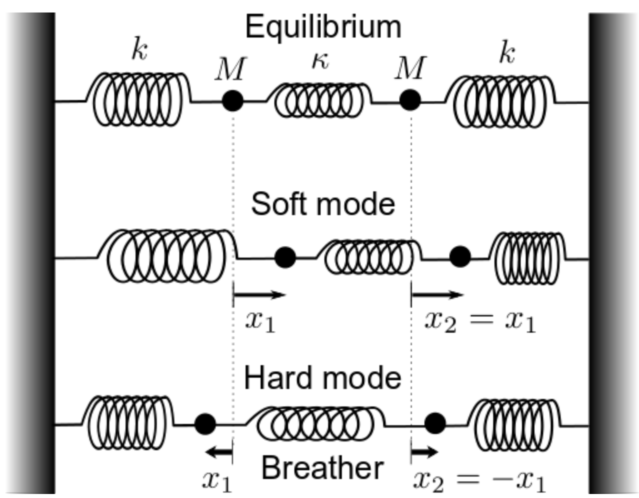
In this example, $\vec{T}_i$ is chosen as a vector with nice integer components. Another nice choice would have been to normalize each $\vec{T}_i$'s so that they become unit vectors. Had we divided each $\vec{T}_i$ by $\sqrt{2}$ following this choice, $\overleftrightarrow{T}$ would have become an orthogonal matrix.

Note that after the generalized eigenvalue problem is solved, each column vector $\vec{T}_i$ tells us about the shape of the i-th normal mode. What does this mean? Suppose that out of all possible normal modes, only η_i is excited. Then, we get $\vec{q} = \eta_i \vec{T}_i$, since $\eta_j = 0$ if $j \neq i$ by assumption and we used Eq. 16.30. Since η_i is just a number, the shape of the normal mode is completely determined by the vector $\vec{T}_i$.

In the current example, $\vec{T}_1$ says that $x_1 = x_2$, which is the description of the shape for this normal mode. In this normal mode, the two masses move exactly in phase. For the second normal mode, $\vec{T}_2$ says that $x_1 = -x_2$, and so the shape of the normal mode is such that the two masses are completely out of phase. If mass 1 is displaced to the right, then mass 2 is displaced to the left by the same amount. If mass 1 is displaced to the left, then mass 2 is displaced to the right by the same amount. Such a mode is called a breathing mode, or a breather.

16.3 The summary of the example

Here is a diagram that summarizes what we found.



Here are some points to think about. They are left for your extra work.

1. From the inspection of the above diagrams *alone*, you can easily get the normal mode frequencies, as well as verifying that the above modes are indeed normal modes (that is, all masses are oscillating at the same frequency). Challenge yourself to demonstrate this fact in a manner that is completely satisfactory to you.

2. The soft mode involves the motion of the center of mass, while the hard mode involves the internal motion with the center of mass fixed. Show that this was to be expected from the Lagrangian point of view—that is, the Lagrangian is separated into that for the center of mass motion and for the internal motion. Demonstrate that separation and show that the two separated parts of the Lagrangian do indeed explain the soft mode and the hard mode completely.

3. What kind of symmetry is responsible for the obvious symmetry properties display by these two modes, namely the symmetric mode (soft mode) and the anti-symmetric mode (hard mode)? [Hint: It is none of the continuous symmetries—the homogeneity or the isotropy of space—that we discussed so far.]

This example is simple enough so that these considerations (in fact, just the last and the first) are sufficient to give all the answers for the normal mode eigenvectors (the shapes of the normal mode, as depicted above) and the normal mode eigenvalues (the frequencies), *without ever using any differential equation or matrix algebra.*