Notes for Lecture 15

Scattering

Here, we describe two particle collisions and then introduce the concept of the scattering crossection. The terms "collision" and "scattering" are used interchangeably in this note.

15.1 Elastic collisions between two particles

In general, two particles scattering off of each other by conserving momentum and angular momentum, assuming that the system is closed.

Even if the system is open, the momentum and the angular momentum would be conserved to a good approximation if the collision occurs in a very short time scale while any external force remains finite. With this in mind, we will assume a closed system throughout this lecture note.

The total kinetic energy, on the other hand, would not be conserved, in general. It will be conserved only for an elastic collision.

15.1.1 General case

For an elastic collision between two particles, we can write as

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \tag{15.1}$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \tag{15.2}$$

Here the subscripts 1 and 2 refer to particle 1 (m_1) and particle 2 (m_2). The symbol u refers to the velocity in the initial state prior to the collision and the symbol v refers to the velocity in the final state after the collision.

Where is the angular momentum conservation rule? At this moment, we do not consider it, assuming that each particle is a point mass interacting only on contact. So, the angular momentum with respect to the contact point is always zero, and so it need not be considered. We will see later in this lecture note that when we relax these assumptions regarding particle size and interaction the angular momentum comes explicitly into play.

However, things are much nicer in the CM reference frame. First of all, one must note that the CM frame moves at a constant velocity since the total momentum $\vec{P} = M\dot{\vec{R}}$ is a conserved quantity. So, if the LAB frame is an inertial frame, then so is the CM frame. Thus, all conservation laws are valid in the CM frame as well. Second, in the CM frame, we will put a prime ($'$) on all vectors, as we did in the last lecture.

$$m_1\vec{u}_1' + m_2\vec{u}_2' = 0 \tag{15.3}$$

$$m_1\vec{v}_1' + m_2\vec{v}_2' = 0 \tag{15.4}$$

$$\frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \tag{15.5}$$

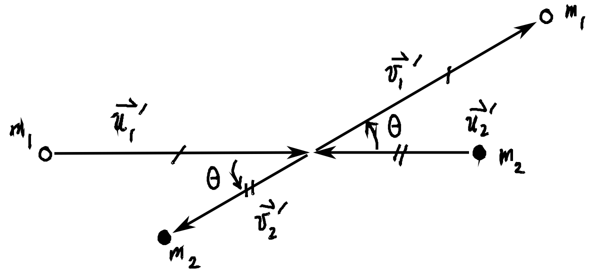
Here, the first two equations follow from the fact that total momentum measured within the CM frame is always zero, by definition (Eq. 14.29).

Thus, in the CM frame, the two vectors $\vec{u}_1'$ and $\vec{u}_2'$ are opposite to each other in direction, as are the two vectors $\vec{v}_1'$ and $\vec{v}_2'$.

What is particularly nice about the CM frame is the fact that if one defines $\vec{p}_{1i} \equiv m_1\vec{u}_1'$ and $\vec{p}_{1f} \equiv m_1\vec{v}_1'$, then the energy conservation equation can be rewritten as $p_{1i}^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right) = p_{1f}^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)$, where $p_{1i} \equiv |\vec{p}_{1i}|$ and $p_{1f} \equiv |\vec{p}_{1f}|$, and where use is made of the fact that $m_1^2u_1'^2 = m_2^2u_2'^2 = p_{1i}^2$ due to $m_1\vec{u}_1' + m_2\vec{u}_2' = 0$ and $m_1^2v_1'^2 = m_2^2v_2'^2 = p_{1f}^2$ due to $m_1\vec{v}_1' + m_2\vec{v}_2' = 0$.

The net result? $p_{1i} = p_{1f}$. That is, the velocity of particle 1 simply changes the direction while its magnitude remains unchanged. There is nothing special about particle 1, and so we expect that the same holds for particle 2 as well. Indeed, it is trivial to see that it does. Namely, the particle 2 also simply deflects in angle with its speed conserved.

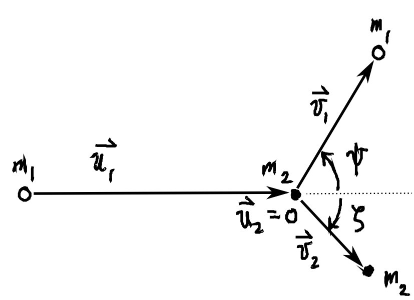
The following diagram is then the most general diagram regarding the elastic collision of two particles as viewed in the CM frame.



The above diagram summarizes the fact that the velocity of each particle is simply deflected while its magnitude is conserved. Note also that this diagram clearly shows that the scattering occurs in a plane, defined by the two directions—the direction of the initial velocity of any particle and the direction of the final velocity of any particle. [Clearly, this fact remains true even for an inelastic scattering.] A curious student may ask "what causes the particles to deflect at an angle θ like drawn here?"—this is an excellent question. If these particles are truly point particles and if they interact only on contact, then all we have is head-on collision ($\theta = \pi$) or no collision ($\theta = 0$). In reality, neither of these assumptions is true and we would need to consider what is called the "impact parameter" as we will do near the end of this lecture note. When we consider it (and thus the angular momentum, which we have not invoked here), we will discover that the deflection angle θ is a function of b (and other constants of motion).

15.1.2 Common case

Commonly, a scattering experiment is performed in such a way that one of the masses is at rest initially. Say, m_2 is initially at rest.



Here, the scattering event is described in the LAB frame, with deflection angles ψ and ζ. In order to figure out the conversion of angles, it is convenient to use the constant total momentum, $P = m_1 u_1$ as the reference scale. In the initial state, particle 1 has momentum $m_1(u_1 - \frac{P}{m_1+m_2}) = P\frac{m_2}{m_1+m_2}$ along the x direction in the CM frame, where we used the fact that the CM frame moves at speed $P/(m_1+m_2)$ along the positive x direction. In the final state, the x component of its linear momentum is given by $P\frac{m_2}{m_1+m_2}\cos\theta$ in the CM frame. In the LAB frame, this becomes $P\frac{m_1+m_2\cos\theta}{m_1+m_2}$.

$$P\frac{m_1+m_2\cos\theta}{m_1+m_2} = p_1\cos\psi. \qquad p_1 \equiv m_1 v_1 \tag{15.6}$$

A similar consideration for particle 2 gives

$$P\frac{m_2(1-\cos\theta)}{m_1+m_2} = p_2\cos\zeta. \qquad p_2 \equiv m_2 v_2 \tag{15.7}$$

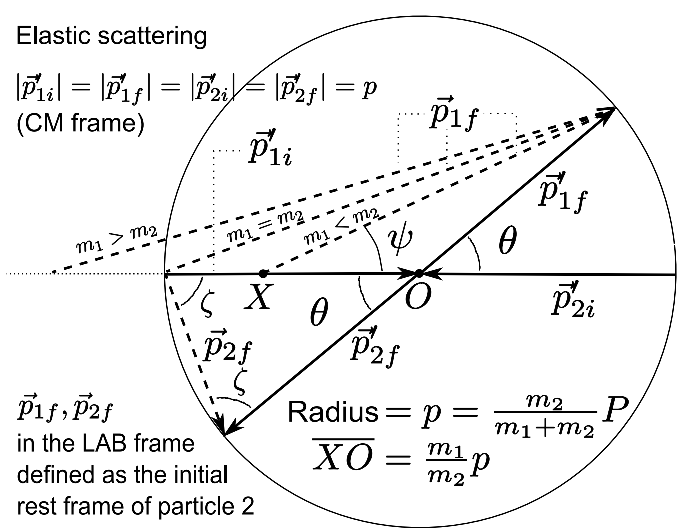
Adding these two equations, we get $P = p_1\cos\psi + p_2\cos\zeta$, which corresponds to the momentum conservation in the x direction. This is as expected. The consideration for the y momentum conservation is easier, since there is no velocity to subtract.

$$P\frac{m_2}{m_1+m_2} = p_1\sin\psi, \tag{15.8}$$
$$P\frac{m_2}{m_1+m_2} = p_2\sin\zeta. \tag{15.9}$$

Dividing Eq. 15.6 by Eq. 15.8, we get the relation between ψ and θ. Dividing Eq. 15.7 by Eq. 15.9, we get the relation between ζ and θ.

$$\tan\psi = \frac{m_2\sin\theta}{m_1+m_2\cos\theta} = \frac{\sin\theta}{m_r+\cos\theta}, \qquad m_r \equiv \frac{m_1}{m_2} \tag{15.10}$$
$$\zeta = \frac{1}{2}(\pi-\theta). \tag{15.11}$$

The geometry of the scattering is conveniently represented in the following diagram made in the momentum plane.



In this diagram, the initial states of the two masses are represented as two horizontal arrows of the same size meeting directed towards each other. The arrows plotted with solid lines represent the momentum values in the CM frame. In the final state, only the direction changes. The magnitude of each of these four arrows is equal to the same value $p = \frac{m_1}{m_1+m_2}P$, as indicated. Apart from the actual value of p, the fact that all four momentum values have the same magnitude p is the general result for any elastic scattering, as we saw in the previous section. Dashed lines represent the momentum values in the LAB frame, which is defined as the initial rest frame of particle 2 in this section. As shown, the final momentum for mass 2 in the LAB frame always connects two points on the circle (of radius p) in such a way to make an isosceles triangle with one side lying horizontal, while that for mass 1 does so only when $m_1 = m_2$.

Note that if $m_2 \gg m_1$, then $\tan\psi \approx \tan\theta$, which means that $\psi = \theta$. This is reasonable, since, in this case, the LAB frame and the CM frame are one and the same, as the CM is fixed at the position of m_2, which will not move.

Another interesting case is $m_1 = m_2$. Using the double angle formulae for sine and cosine, it follows that $\tan\psi = \tan\frac{\theta}{2}$ and therefore $\psi = \theta/2$. In this case, $\psi + \zeta = \pi/2$, which is a well-known fact in a billiard game.

$$\psi = \frac{\theta}{2}, \qquad m_1 = m_2 \tag{15.12}$$

$$\psi + \zeta = \frac{\pi}{2}. \qquad m_1 = m_2 \tag{15.13}$$

15.1.3 Head-on collision

A head-on collision corresponds to the case $\theta = \pi$. Note that the $\theta = 0$ case is *not* a head-on collision. If there is any collision, then in the CM frame, the velocities must change the direction. The $\theta = 0$ case corresponds to no collision at all.

For a head-on elastic collision, we have a one dimensional collision problem. Let us use u_1', u_2' as vector quantities (i.e. signed numbers).

The properties that we found above for an elastic collision can be written for head-on collisions as

$$u_1' = -v_1' \tag{15.14}$$

$$u_2' = -v_2' \tag{15.15}$$

From this it is trivial to derive the following result.

Relative velocity flip in a head-on elastic collision

In the CM frame,

$$u_1' - u_2' = v_2' - v_1'. \tag{15.16}$$

This immediately implies that in *any* frame

$$u_1 - u_2 = v_2 - v_1. \tag{15.17}$$

This is an extremely useful relation to remember for a head-on elastic collision. Instead of using momentum conservation plus kinetic energy conservation, it is much more convenient to use momentum conservation plus this "relative velocity flip" equation.

Example: If $u_2 = 0$, then we have

$$m_1 u_1 = m_1 v_1 + m_2 v_2 \tag{15.18}$$

$$u_1 = v_2 - v_1 \tag{15.19}$$

$$\tag{15.20}$$

Solving for v_1 and v_2, we get

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \tag{15.21}$$

$$v_2 = \frac{2m_1}{m_1 + m_2} u_1 \tag{15.22}$$

If $m_1 \gg m_2$, then $v_1 \approx u_1$ and $v_2 \approx 2u_1$. This is the case when $\psi \approx 0$ and $\zeta \approx 0$. If $m_1 \ll m_2$, then $v_1 \approx -u_1$ and $v_2 \approx 0$. This is the case when $\psi \approx \pi$ and $\zeta \approx 0$. If $m_1 = m_2$, then we get $v_1 = 0$ and $v_2 = u_1$ (if you are a billiard player, you probably know that this happens with no English and head-on collision). Note that the solution in the previous section for the equal mass case, $\psi + \zeta = \pi/2$, implies $\psi = \pi/2$ since $\zeta = 0$ obviously. This might seem like a paradox, since how can ψ be $\pi/2$ if we are considering a collision in one dimensions? But, as a matter of fact, ψ is indeterminate, since $v_1 = 0$.

15.2 Inelastic collision between two particles

Collisions between two macroscopic objects tend to be inelastic. In this case, the kinetic energy conservation cannot be used.

The **coefficient of restitution** measures the degree of inelasticity.

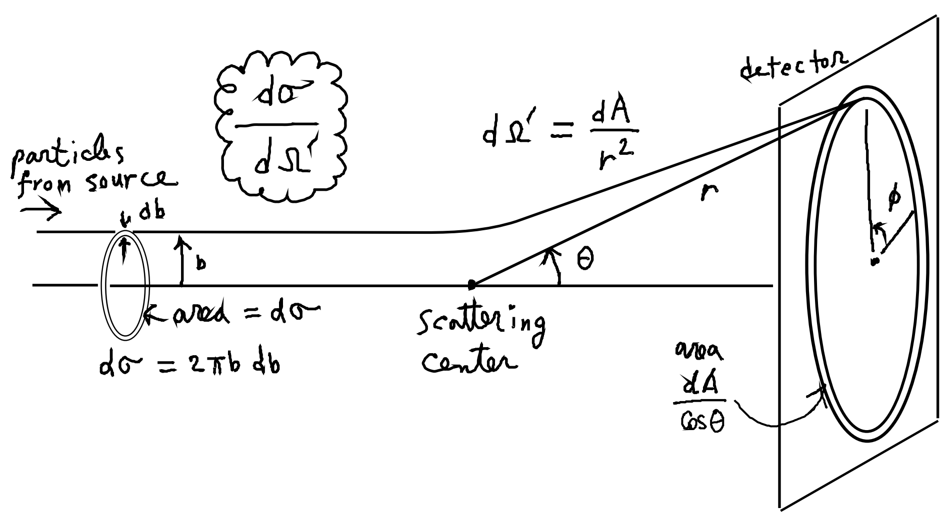
Consider a head-on collision. The coefficient of restitution, ε, is defined as

$$\varepsilon = \frac{|v_1 - v_2|}{|u_1 - u_2|}. \tag{15.23}$$

Note that $\varepsilon = 1$ only for an elastic scattering, for which we have shown in the last section, $v_2 - v_1 = u_1 - u_2$. The $\varepsilon = 0$ case is the **totally inelastic case**: this occurs when two bodies become one. One can show that for a general inelastic collision $\varepsilon < 1$.

15.3 Impact parameter, differential crossection

Consider the following diagram. To be specific, we consider a central force problem where the potential function is a function of the relative coordinate vector, $r = |\vec{r}_1 - \vec{r}_2|$. This diagram is drawn with the origin (the "scattering center") at the zero of the $\vec{r}$ coordinate corresponding to the "internal Lagrangian" L_i (or L_{int}).



This diagram depicts a situation where particles from a source are incident on a scattering center and get deflected. The incident beam of particles is assumed to have a fixed direction of velocity. This direction defines the horizontal axis of the diagram. At some distance from the scattering center, a detector measures the number of particles that hit it.

Because we are considering this problem in the $\vec{r}$ coordinate system, note that the "particle" here is the particle with the reduced mass μ.

The particle source and the particle detector are both very far away from the scattering center. In comparison b is a very small number. However, due to space limitation, this diagram is not up to scale in this regard.

In the above diagram, we are assuming that the potential felt by the particle is repulsive. One can draw an equivalent diagram for an attractive potential case as well.

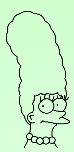
What is the meaning of the **impact parameter** b? (1) Simply speaking it is the lateral distance between the scattering center and the initial beam, as depicted above. (2) More physically speaking, note that we are using a relative coordinate here, and, thus, in fact the "scattering center" can be considered as the position of m_2. So, b is the lateral distance between m_1 and m_2, in the initial state, i.e. when $\vec{r}_1$ is at the far left. $\vec{r} = \vec{r}_1 - \vec{r}_2$. (3) Note that the **angular momentum** for the motion is given by $\mu u b$, where u is the relative velocity in the initial state ($u = |\dot{\vec{r}}|$ when the particle emerges from the far left) and μ is the reduced mass. We know, from previous lecture, this angular momentum to be the angular momentum associated with the Lagrangian L_i. From Eqs. 14.32, we can also interpret this angular momentum as the **angular momentum as measured in the CM frame**. (4) If we assume a common scattering experiment where m_2 is initially at rest, then $u = u_1$, and we see that b can be interpreted as the angular momentum $\mu u_1 b$ up to the factor μu_1.

What is the meaning of the θ value? Is it equal to θ as defined in page 3? **The answer is yes!** Please examine the figure of page 3, and see that it is clear that θ is also the deflection of the relative velocity due to the scattering, not just the deflection of the velocity in the CM frame.

The key observation is that b and θ has a functional relationship. **This is why particles deflect on collision.** Typically but not always, as b increases, θ will tend to decrease (this is why the outer circle on the left side—the source side—is connected to the inner circle on the right side—the detector side: by the same token, the inner circle on the left side corresponds to the outer circle on the right side). In other words, if the particle goes close to the scattering center, it will experience a greater amount of force, and thus a greater amount of momentum change and thus a greater amount of deflection.

Note that in general we expect a rotational symmetry, by the isotropy of space. This means in particular that the scattering process that we depicted here will be independent of the angular position, ϕ, around the circle.

The diagram of in page 7 is a very important schematic diagram of a typical scattering experiment. And, please keep in mind that many physics experiments are scattering experiments. We poke and break to figure things out—and many of our poking and breaking activities are scattering experiments, whether it is of smashing particles at the LHC or the electron spectroscopy at the SSRL. Of course, details of scattering experiments can differ vastly depending on the type of physics being explored, but all scattering experiments measures the number of output particles, i.e. scattered particles, per given amount of input particles. This is what this section is about.



Differential crossection

The quantity

$$\frac{d\sigma}{d\Omega'}$$

i.e., the ratio of the area on the left side to the ratio of the solid angle on the right side, is defined as the **differential crossection**.
Its **more physical meaning** is

$$\frac{d\sigma}{d\Omega'} = \frac{dN/d\Omega'}{I}$$

where I is the flux density of incident particles (the number of incident particles per unit area per unit time) and $dN/d\Omega'$ is the number of particles detected per solid angle per unit time. So, the differential crossection is proportional to the probability of a particle to be scattered to a certain solid angle, and has the dimension of area.

From the figure of page 7, it can be easily seen why $\frac{d\sigma}{d\Omega'} = \frac{dN/d\Omega'}{I}$. First, let us note that $dN = I d\sigma$. Second, divide both sides by $I d\Omega'$.

Note that the solid angle concept enters since the number of particles detected by the detector will decrease as $1/r^2$ as a function of r.

Note that $d\sigma = 2\pi b db$ (the area of the ring on the left side of the figure) and $d\Omega' = 2\pi \sin\theta d\theta$ (this can be derived from dA/r^2). Thus, in terms of b and θ, we get

$$\frac{d\sigma}{d\Omega'} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{15.24}$$

Here, we put $db/d\theta$ in the absolute sign, since it is generally negative, and we define $d\sigma/d\Omega'$ as a positive quantity. Using this formula, the differential crossection can be calculated for a given interaction potential.

Finally, note that the scattering crossection has been discussed in the $\vec{r}$ coordinate system so far. As the solid angle in this coordinate system is identical with the solid angle in the CM frame, we used the notation $d\Omega'$ for the solid angle. Of course, unless $m_2 \gg m_1$, it is necessary to convert to the angle ψ for the crossection to compare theory with experiment, which occurs in the LAB frame. The differential crossection in the LAB frame can be written as $d\sigma/d\Omega$. The conversion is done with the aid of the kinematic relationship between ψ and θ as we discussed in Section 15.1.2. To carry out such a conversion note that $d\sigma$ is an invariant quantity between the two frames, since $d\sigma = 2\pi b db$.

$$\frac{d\sigma}{d\Omega'}d\Omega' = \frac{d\sigma}{d\Omega}d\Omega. \tag{15.25}$$

Here, $d\Omega = 2\pi\sin\psi d\psi$ and $d\Omega' = 2\pi\sin\theta d\theta$. Thus, we get

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega'}\frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi}. \tag{15.26}$$

As the relation between ψ and θ are known, the conversion factor here can be calculated. The details are left out and we present only the final result.

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega'}\frac{\left(m_r\cos\psi + \sqrt{1-m_r^2\sin^2\psi}\right)^2}{\sqrt{1-m_r^2\sin^2\psi}}. \tag{15.27}$$

Example. For the potential given by k/r, the following result can be obtained (Rutherford scattering; see other textbooks such as Gregory or Thornton-Marion) by integrating $\theta(r)$ given the initial energy and the impact parameter b:

$$\frac{d\sigma}{d\Omega'} = \frac{k^2}{(4T_0')^2}\frac{1}{\sin^4(\theta/2)} \tag{15.28}$$

where T_0' is the kinetic energy of the initial state in the CM frame. Note that this crossection remains identical whether the interaction is attractive or repulsive, i.e., it is independent of the sign of k. It is a remarkable fact that this Rutherford formula remains valid even in (the leading order perturbation theory of) quantum mechanics. In general, scattering crossections of fundamental particles must be calculated by quantum mechanics.