Notes for Lecture 14

Kepler problem and many-body problem

Here, we will wrap up our discussions on the Kepler problem and then move on to the more general many body problem.

14.1 Elliptical orbits

14.1.1 Turning points, angular momentum conservation

Let us come back to the discussion of elliptical orbits, applicable to comets as well as planets. In an elliptical orbit,

$$r_{min} = \frac{\alpha}{1+\varepsilon}, \quad r_{max} = \frac{\alpha}{1-\varepsilon}. \tag{14.1}$$

These are "turning points" for the r motion, or **apsides**. In particular, these are called **pericenter** (r_{min}), or **apocenter** (r_{max}). Think "apex" to remember which is which. For objects orbiting around the Earth, we talk about perigee and apogee, and for objects orbiting around the Sun, we talk about perihelion and aphelion. Note that **the speed is maximum (minimum) at** r_{min} **(**r_{max}**)**. This follows directly from the energy conservation $E = K + U = \frac{1}{2}\mu v^2 - \frac{k}{r}$. Call them v_{min} and v_{max}. At the turning points, $\vec{v} \perp \vec{r}$, and so the angular momentum conservation means for apsides,

$$r_{min} v_{max} = r_{max} v_{min}. \tag{14.2}$$

14.1.2 Kepler's third law

The total area of an ellipse is πab. As we saw, the areal velocity is constant **for any central force**, and is given by $dA/dt = l/2\mu$. This proves Kepler's second law (this is no news to us; we did this in the previous lecture). Now, the period τ for an elliptical motion is given by $\tau = \pi ab/(dA/dt) = 2\pi ab\mu/l$. As $b = \sqrt{\alpha}\sqrt{a} = l\sqrt{a}/\sqrt{\mu k}$ (Eqs. 13.36, 13.37, 13.34), **we get the following Kepler's third law**

$$\boxed{\tau^2 = \frac{4\pi^2\mu}{k}a^3. \qquad (14.3)}$$

Qualitatively speaking, this law should read generally thus: "the period squared is proportional to the linear dimension of the orbit cubed." For instance, if we had eliminated a instead of b, then we would have gotten $\tau^2 \propto b^3$ with a different proportionality constant. Or, one could say that the average radius of an elliptical orbit is proportional to $\tau^{2/3}$. However, as it is written, the above form, expressed in terms of a, has a certain appeal, as we shall see shortly. Likewise, note that

$$E = -\frac{k}{2a} \qquad (14.4)$$

for an elliptical orbit (Eq. 13.36; note $E < 0$). These relations are worth remembering, which is possible without too much effort (see the box below).

14.1.3 Virial theorem

The "virial theorem" for power law forces is a powerful theorem that can be handy very often. It is useful for describing planetary motions. Furthermore, when classical mechanics must be replaced by quantum mechanics, this theorem survives: so the theorem remains valid for electron motion in an atom. Here, we will not be distracted by the derivation of the theorem (this is left for your optional extra work; I suggest some on-line searching, if you are interested), but simply state the most useful end-result as a theorem:

$$\langle K\rangle = \frac{n}{2}\langle U\rangle \qquad \text{if } U \propto r^n \qquad (14.5)$$

where $r = |\vec{r}|$.

How to remember, or quickly re-derive, stuff?

Suppose you need to remember the above Kepler's third law (and some other crucial stuff that we derived above). **You can do it, very easily!** Do you have to memorize it? There is no way *I* can memorize such a formula! But, wait, there is a clever way to "recall," without doing all the complicated stuff that we just did! Just remember this one first: **the uniform circular motion is your best friend. . .** For the circular motion, we have

$$\frac{\mu v^2}{r} = \frac{k}{r^2} \qquad \text{Centripetal force equation} \tag{14.6}$$

$$E = K + U = \frac{\mu v^2}{2} - \frac{k}{r} \qquad \text{Energy conservation} \tag{14.7}$$

where r, v, K, U are all *constants*. By expressing the centripetal force equation in terms of τ, by noting $v = r\omega = r2\pi/\tau$, we get

$$\tau^2 = \frac{4\pi^2\mu}{k} r^3 \qquad \textbf{Kepler's 3rd law for circular orbit} \tag{14.8}$$

By multiplying the first equation by r, we get

$$-2K = U \qquad \textbf{Virial theorem example} \tag{14.9}$$

$$E = -T = \frac{U}{2} = -\frac{k}{2r} \tag{14.10}$$

The good news is that the last three equations remain valid even for elliptical orbits, if we make the following substitutions: ($\langle\cdots\rangle_\tau$: average over a period)

$$r \to a,\ K \to \langle K\rangle_\tau,\ U \to \langle U\rangle_\tau$$

For both circular orbits and elliptical orbits, we have:

$$\tau^2 = \frac{4\pi^2\mu}{k} a^3$$

$$\langle U\rangle_\tau = -2\langle K\rangle_\tau$$

$$E = -\langle K\rangle_\tau = \frac{\langle U\rangle_\tau}{2} = -\frac{k}{2a}$$

The angular momentum conservation, $l = \mu r^2\dot{\theta} = 2\mu dA/dt$ (valid for *any* central force) and these three equations are **crucial things to rather easily recall** as shown here (not memorize!).

Here, the notation $\langle Q\rangle$ means the following:

$$\langle Q\rangle \equiv \frac{1}{T}\int_0^T dt\, Q \tag{14.11}$$

where $T = \tau$ (the period of motion, for a periodic motion) or $T \to \infty$ (for a non-periodic bound motion). Let us talk specifically about the motion under Hooke's law force ($n = 2$) or the motion under Newton's law of gravity ($n = -1$). The motion under Hooke's law force is always a bound motion, and it turns out that it is always periodic. As we saw in the last lecture, the motion under Newton's law of gravity can be bound ($0 \le \varepsilon < 1$) or unbound ($\varepsilon \ge 1$). For the discussion of the virial theorem, we focus on the bound motion only. In this case, the motion is also a periodic motion, tracing an ellipse. So, we will use $\langle Q\rangle$ in the first meaning, the time average over a period, in the rest of this section, and use a subscript τ to make it clear that we are doing so, as in $\langle Q\rangle_\tau$.

For the Hooke's law force, we get $\langle K\rangle_\tau = \langle U\rangle_\tau$ from the virial theorem. This fact is also easy to prove explicitly, if we note that the average value of sine or cosine squared is $1/2$, if the average is taken over any multiple of $\tau/2$, and that K and U are proportional to sine squared or cosine squared, with the same proportionality constant between K and U. Let us now consider the current Kepler problem, $n = -1$. According to this theorem, we have, for any elliptical orbit,

$$2\langle K\rangle_\tau = -\langle U\rangle_\tau. \tag{14.12}$$

Since the energy $E = K + U$ is constant, it then follows that

$$E = -\langle K\rangle_\tau = \frac{1}{2}\langle U\rangle_\tau. \tag{14.13}$$

Since $E = -\frac{k}{2a}$ (Eq. 14.4), this means that

$$\langle U\rangle_\tau = -\frac{k}{a}, \tag{14.14}$$

$$\langle K\rangle_\tau = \frac{k}{2a}. \tag{14.15}$$

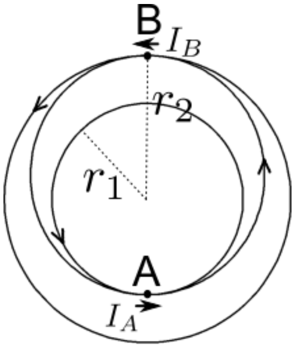
The first equation means that

$$\left\langle \frac{1}{r}\right\rangle_\tau = \frac{1}{a}. \tag{14.16}$$

If you like, you can verify this using the solution obtained in the last lecture. This is left for your optional extra work (Hints: the time integral can be turned into the angle integral using the angular momentum conservation, and then you can use the orbital formula, $\frac{\alpha}{r} = 1 + \varepsilon\cos\theta$).

14.1.4 Hohmann transfer

Suppose that we like to travel to Mars. We like to do it in a fuel efficient way. How may we accomplish this goal? The most fuel efficient way to transfer from one circular orbit to another circular orbit under the central potential $U = -\frac{k}{r}$, where $k = GMm$, is known to be the Hohmann transfer. So, let us consider travelling to the Mars orbit using this method. In this process, two quick forward thrusts are applied to a spaceship. Here, "quick" means "quick enough so that the position of the spaceship can be considered unchanging." The first forward thrust (applied at point A in the diagram) causes the spaceship to deviate from the Earth's circular orbit, and follow an elliptical orbit that is tangential to both the Earth orbit and the Mars orbit around the Sun. The second forward thrust (applied at point B in the diagram) is applied just when the spaceship enters the circular orbit of Mars.



Note that when we go from the small Earth orbit (radius r_1) to the large Mars orbit (radius r_2), the speed of the spaceship ends up decreasing overall, since the kinetic energy is less for a larger orbit (Eq. 14.15 with $a = r$ (radius) for a circular motion). Then, it may seem paradoxical that we applied two quick *forward* thrusts to increase the kinetic energy at each time when we applied thrust. By accelerating twice, we go from a small circular path (with v_1) to a larger circular path (with v_2), but yet, we must have $v_2 < v_1$.

The paradox is resolved by the fact that the speed decreases from point A to point B while the spaceship is on the elliptical orbit, and that the amount of this speed decrease is greater than the total increase of the speed by the forward thrusts.

At point A, the speed of the spaceship increases from v_1 to v_i by impulse (see Section 14.2.2) I_A: $I_A = F_A \Delta t = mv_i - mv_1$, where F_A is the thrust. At point B, similarly, $I_B = mv_2 - mv_f$. Here, we defined v_i as the speed at point A for the elliptical orbit, and v_f as the speed at point B for the elliptical orbit. We also assumed that the mass of the spaceship does not change significantly after each thrust.

We should be able to express v_i and v_f in terms of v_1 and v_2. Let us see how we can do so. First, let us note the following identities.

$$r_1 v_1^2 = r_2 v_2^2 = \frac{k}{m}, \qquad \text{viriral theorem for circular orbits} \qquad (14.17)$$

$$r_1 v_i = r_2 v_f, \qquad \text{angular momentum conservation, ellptical orbit} \qquad (14.18)$$

$$\frac{1}{2} m v_i^2 - \frac{k}{r_1} = \frac{1}{2} m v_f^2 - \frac{k}{r_2}. \qquad \text{energy conservation, ellptical orbit} \qquad (14.19)$$

Using the first equation, the last equation can be written as $\frac{1}{2}mv_i^2 - mv_1^2 = \frac{1}{2}mv_f^2 - mv_2^2$. This equation can be rewritten as

$$v_i^2 - 2v_1^2 = v_f^2 - 2v_2^2. \tag{14.20}$$

This equation and Eq. 14.18 can be used to solve for v_i and v_f.

$$v_i = \sqrt{\frac{2r_2}{r_1 + r_2}}\, v_1, \tag{14.21}$$

$$v_f = \sqrt{\frac{2r_1}{r_1 + r_2}}\, v_2. \tag{14.22}$$

Using Eq. 14.17, we can rewrite these purely in terms of v_1 and v_2

$$v_i = \frac{\sqrt{2}\, v_1^2}{\sqrt{v_1^2 + v_2^2}}, \tag{14.23}$$

$$v_f = \frac{\sqrt{2}\, v_2^2}{\sqrt{v_1^2 + v_2^2}}. \tag{14.24}$$

Let us ask how much time will it take for a one-way Hohmann transfer? The required time is the half the period for the elliptical orbit, and it is given by (taking $\tau/2$ from Eq. 14.3 and noting that $\mu \approx m$ and $a = \frac{r_1+r_2}{2}$)

$$\pi\sqrt{\frac{m}{k}}\left(\frac{r_1 + r_2}{2}\right)^{3/2} = \pi\sqrt{\frac{1}{GM}}\left(\frac{r_1 + r_2}{2}\right)^{3/2}. \tag{14.25}$$

Using the data for Sun's mass, the Earth orbit size, and the Mars orbit size, this time is about 260 days. Clearly, the same principle can be used for satellite maneuvers around the Earth.

Now, what if the spaceship is to be brought back to a small circular orbit from a large circular orbit? Would the Hohmann transfer still remain as the most fuel-efficient way? The answer is yes since the system is time-reversal invariant. Please do not take these words for granted. Fill in the blanks and establish your own satisfactory logic to see that the answer is indeed yes.

14.2 Many body system

Let us consider a system of many particles, not just two.

Any classical mechanics object is a many body system by definition, as we noted in Lecture 1. So, here we make a sort of a full circle. Nevertheless, it is important to recognize certain key concepts such as the center of mass.

Indeed, we have already recognized this in a previous lecture, in Section 13.2, where properties for any many-body system had been already discussed.

To start, let us assume that we have a system of particles, m_α, where α is the index for each particle, which we describe in an inertial frame. Let us define this inertial frame as our "LAB" frame. In this LAB frame, let $\vec{r}_\alpha$ be the position vector for m_α.

14.2.1 Center of mass

The center of mass coordinate $\vec{R}$ is defined as

$$\vec{R} \equiv \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{M} \tag{14.26}$$

where

$$M \equiv \sum_\alpha m_\alpha \tag{14.27}$$

is the total mass.

We now define the "CM" (center of mass) frame as the reference frame for which the origin is at $\vec{R}$. Note that the CM frame is generally *not* an inertial frame, as it is, in general, accelerating relative to the LAB frame, which we assumed to be an inertial frame.

Let the position vectors measured in the CM frame be $\vec{r}'_\alpha$.

By construction, then, we have $\vec{r}'_\alpha = \vec{r}_\alpha - \vec{R}$, or

$$\vec{r}_\alpha = \vec{R} + \vec{r}'_\alpha. \tag{14.28}$$

Multiplying this equation by m_α, and then summing up over α, we get a trivial but pretty important result:

$$0 = \sum_\alpha m_\alpha \vec{r}'_\alpha. \tag{14.29}$$

This is a trivial statement, since what it says is that the center of mass in the CM frame is zero. Of course, that follows how we defined the CM frame! Taking the time derivative, we get

$$\sum_\alpha m_\alpha \dot{\vec{r}}'_\alpha = 0. \tag{14.30}$$

These two identities can be used to prove the following extremely useful kinematics relations.

$$\vec{P}_{tot} \overset{def}{=} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}, \tag{14.31}$$

$$\vec{L}_{tot} \overset{def}{=} \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = M\vec{R}\times\dot{\vec{R}} + \sum_\alpha m_\alpha \vec{r}\,'_\alpha \times \dot{\vec{r}}\,'_\alpha, \tag{14.32}$$

$$T_{tot} \overset{def}{=} \frac{1}{2}\sum_\alpha m_\alpha |\dot{\vec{r}}_\alpha|^2 = \frac{1}{2}M|\dot{\vec{R}}|^2 + \frac{1}{2}\sum_\alpha m_\alpha |\dot{\vec{r}}\,'_\alpha|^2. \tag{14.33}$$

Proving these identities is left for your exercise.

These identifies show the importance of the center of mass. All of these kinematical quantities are nicely split into two terms, the center of mass term with the mass M, and the internal term as measured in the CM frame. For the linear momentum the latter is zero by definition.

The above equations refer to the kinematics only. How about the dynamics?

$$\dot{\vec{P}}_{tot} = \vec{F}^{(e)} \overset{def}{=} \sum_\alpha \vec{F}^{(e)}_\alpha, \tag{14.34}$$

$$\dot{\vec{L}}_{tot} = \vec{N}^{(e)} \overset{def}{=} \sum_\alpha \vec{r}_\alpha \times \vec{F}^{(e)}_\alpha. \tag{14.35}$$

Here, $\vec{F}^{(e)}_\alpha$ is the total *external* force that applies to m_α. And, $\vec{N}^{(e)}$ is the total torque due to external forces. For sure, note that there are internal forces that apply to m_α as well, due to all other masses m_β with $\beta \neq \alpha$. However, due to the symmetry of space, the total momentum must be conserved for a closed system, and so is the total angular momentum. This is the reason why in these expressions only external forces appear, since without external forces (closed system)

$$\dot{\vec{P}}_{tot} = 0, \quad \dot{\vec{L}}_{tot} = 0. \qquad \text{closed system} \tag{14.36}$$

Note that the following equations hold generally.

$$\dot{\vec{L}}_M = \frac{d}{dt}\left(M\vec{R}\times\dot{\vec{R}}\right) \qquad \vec{L}_M \equiv M\vec{R}\times\dot{\vec{R}} \tag{14.37}$$

$$= \vec{R}\times\vec{F}^{(e)} \qquad \text{using } \dot{\vec{R}}\times\dot{\vec{R}} = 0 \text{ and Eq. 14.34} \tag{14.38}$$

$$\dot{\vec{L}}' = \frac{d}{dt}\left(\sum_\alpha m_\alpha \vec{r}\,'_\alpha \times \dot{\vec{r}}\,'_\alpha\right) \qquad \vec{L}' \equiv \sum_\alpha m_\alpha \vec{r}\,'_\alpha \times \dot{\vec{r}}\,'_\alpha \tag{14.39}$$

$$= \vec{N}^{(e)\prime} \overset{def}{=} \sum_\alpha \vec{r}\,'_\alpha \times \vec{F}^{(e)}_\alpha \qquad \text{proof left for your optional extra work} \tag{14.40}$$

where $\vec{L}_{tot} = \vec{L}_M + \vec{L}'$, the sum over the angular momentum of the mass M at the center of mass coordinate and the angular momentum measured within the CM frame. $\vec{N}^{(e)\prime}$ is the torque around the center of mass due to all external forces.

Notice that the result $\dot{\vec{L}}' = \vec{N}^{(e)\prime}$ is a highly non-trivial one, since the reference frame, the CM reference frame, is a non-inertial one. It is a testament to the special nature of the CM reference frame.

Lastly, the dynamical equation $\dot{\vec{P}}_{tot} = \vec{F}^{(e)}$ (and $\dot{\vec{L}}_M = \vec{R} \times \vec{F}^{(e)}$) can be seen as none other than Newton's second law for a "point particle," i.e., the equation from which we started all of these discussions!

14.2.2 Collisions

A collision generally refers to a process of a short lived process by which two objects come into contact and exchange forces in a dramatic manner.

The exchange of forces is conveniently quantified by the impulse.

$$\vec{I} = \int_{t_1}^{t_2} dt \vec{F}(t) \tag{14.41}$$

where $\vec{F}(t)$ is a force that is experienced by one object during the collision process. Putting $\Delta t \equiv t_2 - t_1$, we get

$$\vec{I} = \vec{F}_{ave} \Delta t. \tag{14.42}$$

Usually, Δt is small. Even for a small period of time, the impulse is finite, due to a large force $\vec{F}_{ave}$ exchanged. For the object that receives the impulse, its momentum change during Δt is given by

$$\Delta \vec{P} = \vec{I}. \tag{14.43}$$

Put another way, if Δt is short, and if $\vec{F}_{ave}$ is finite, then the impulse is small, and the effect of collision is negligible. This is the case of the "pulling the table cloth under the cups and plates" trick, sometimes played by waiters at a restaurant.

For collisions occurring in a closed system, the angular momentum and the momentum are always conserved. However, even for a closed system, a collision is generally inelastic—the initial total kinetic energy is not quite equal to the final total kinetic energy as some of the energy goes into modifying the internal state of matter ("wear and tear"). Often, however, a collision can be approximated well as an elastic collision. Even for a non-closed system, either the angular momentum or the momentum is often conserved along certain directions. In these cases, a close inspection of the system in terms of external torques or forces is required to make such a conclusion, while it is often the case that which quantity is conserved is rather obvious. However, as a general rule, if the collision between macroscopic objects is not said to be elastic, then it is most likely inelastic—so one is advised *not* to assume an elastic collision without any good reason.