Notes for Lecture 13

Central force, the Kepler problem

Central force problems are important. All planet motion problems are of this kind to a good approximation. Also, crude but useful classical mechanics models of atoms and nucleons are of this kind. Crude, because classical mechanics fails for atoms and nucleons, but useful, because sometimes a good way to grasp the results of quantum mechanics is to start from classical mechanics and then to add some quantum mechanical thoughts to it. Although such a process is generally a dangerous one, since quantum mechanics is very non-intuitive and very different from classical mechanics, our knowledge of some things such as symmetry and conservation are completely transferable even to the quantum regime.

In setting up the central force problem, we will emphasize the principles of symmetry and conservation.

13.1 Two-body problem—Separation of variables

Before we set up a central force problem, we will step back a bit and consider a slightly more general two body problem. The reason will be clearly explained within this section and the next.

Let us consider a two body problem with the potential that depends on the relative coordinate $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$.

$$L = \frac{1}{2}\left(m_1|\dot{\vec{r}}_1|^2 + m_2|\dot{\vec{r}}_2|^2\right) - U(\vec{r}). \tag{13.1}$$

If U depended only on the *magnitude* of $\vec{r}$, then we would have a central force problem. But here we leave U as slightly more general. The total number of degrees of freedom is six, three from $\vec{r}_1$ and three from $\vec{r}_2$. A different way to distribute the degrees of freedom is to define

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}, \tag{13.2}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2. \tag{13.3}$$

Here, $M \equiv m_1 + m_2$, $\vec{R}$ is the **center of mass coordinate**, and $\vec{r}$ is the **relative coordinate**. The inverse transformation is, after some algebra,

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}, \tag{13.4}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}. \tag{13.5}$$

From this, one obtains

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}, \tag{13.6}$$

$$\dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}. \tag{13.7}$$

Plugging this into the above Lagrangian, we get

$$L = \frac{1}{2}M|\dot{\vec{R}}|^2 + \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(\vec{r}) \tag{13.8}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad \text{the } \textbf{reduced mass}. \tag{13.9}$$

Re-expressing the Lagrangian in terms of $\vec{R}$, $\vec{r}$, and their time derivatives was a very nice thing to do, since the two sets of degrees of freedom for $\vec{R}$ and $\vec{r}$ are now nicely separated: we achieved the **separation of variables**.

$$L = L_{cm} + L_i, \tag{13.10}$$

$$L_{cm} = \frac{1}{2}M|\dot{\vec{R}}|^2, \qquad \text{the motion of center of mass} \tag{13.11}$$

$$L_i = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(\vec{r}). \qquad \text{the internal motion} \tag{13.12}$$

As indicated here, L_{cm} represents the motion of the center of mass, while L_i represents the internal motion of this two-body system.

It can be seen that L_{cm} is a highly symmetric Lagrangian. It is invariant both translationally and rotationally. So, the linear momentum of the center of mass, $M\dot{\vec{R}}$, and the angular momentum of the center of mass, $M\vec{R} \times \dot{\vec{R}}$, are conserved. Also, the energy, $\frac{1}{2}M|\dot{\vec{R}}|^2$, is conserved.

Are any of these conserved quantities for L_{cm} correspond to any conserved quantity of the whole system? The answer is yes for the linear momentum, but no for all other quantities.

That the total linear momentum of the system is equal to $M\dot{\vec{R}}$ can be shown in many ways. The most straightforward way may be using the general definition of the momentum (Eqs. 9.39 and 9.43):

$$\vec{p}_{\vec{R}} = \frac{\partial L}{\partial \dot{\vec{R}}} = \frac{\partial L_{cm}}{\partial \dot{\vec{R}}} = M\dot{\vec{R}}. \qquad (13.13)$$

Note that the notation $\frac{\partial}{\partial \dot{\vec{R}}}$ means the gradient operation with respect to the velocity vector $\dot{\vec{R}}$ (see Eqs. 9.37 and 9.35) and here we have made use of the fact that L_{cm} depends on $\dot{\vec{R}}$ but L_i does not.

13.2 Symmetry considerations

Now is a good time to remind ourselves of the following considerations of the basic symmetry principles (LN 9). It would be important for you to know that they are valid for any physical systems.

Consider an arbitrary *closed* system consisting of many constituent particles. Then, the *total* linear momentum, the *total* angular momentum, and the *total* energy are conserved, by the homogeneity of space, the isotropy of space and the homogeneity of time, respectively.

In classical mechanics[1], the total linear momentum is carried by the motion of the center of mass alone, no matter how many particles are considered. This has been demonstrated explicitly in the previous section for a simple two body problem, and it can be generalized to a many-body case (left for your optional work).

The cases of the total angular momentum and the total energy are quite different matters. Namely, internal motions contribute to these quantities, conserved for a closed system.

[1]As always, this means classical mechanics as treated in this course, excluding relativistic mechanics.

So, while the Lagrangian for the motion of the center of mass, L_{cm}, is always simply given by Eq. 13.11 for a closed system, that simple Lagrangian accounts only for the total linear momentum but not for the total angular momentum or the total energy. Note also that the Lagrangian L_{cm} given by Eq. 13.11 for a closed system simply represents Newton's first law.

13.2.1 Linear momentum conservation

As we just discussed, for any closed many particle system, the linear momentum associated with $\vec{R}$ in L_{cm} (Eq. 13.11) accounts for the *total* linear momentum, which is conserved.

Consider the following point. In the two body problem defined in Section 13.1, would we have ended up with the separation of variables for $\vec{R}$ and $\vec{r}$, had the potential energy been a more general function such as $U(\vec{r}_1, \vec{r}_2)$? For example, we could dream up functions like $U(\vec{r}_1, \vec{r}_2) = A\vec{r}_1 \cdot \vec{r}_2$ or $U(\vec{r}_1, \vec{r}_2) = B|\vec{r}_1^{\,2} - \vec{r}_2^{\,2}|$. These functions seem OK, since they are scalar quantities. They also are invariant when $\vec{r}_1$ and $\vec{r}_2$, which is what we would expect for two interacting particles[2]. However, they are *not* valid potential energies, since they do not respect the translational invariance. In other words, if two particles are interacting with each other and nothing else, then the two particles form a closed system: then, by the homogeneity of space, the Lagrangian must be invariant under the translation $\vec{r}_1 \to \vec{r}_1 + \delta\vec{r}$ and $\vec{r}_2 \to \vec{r}_2 + \delta\vec{r}$, for any arbitrary constant shift $\delta\vec{r}$. Functions such as $\vec{r}_1 \cdot \vec{r}_2$ and $|\vec{r}_1^{\,2} - \vec{r}_2^{\,2}|$ are *not* invariant by such transformation.

The most general form of a function $U(\vec{r}_1, \vec{r}_2)$, which is translationally invariant, is $U(\vec{r}_1 - \vec{r}_2)$. This is the reason why we considered this particular form of potential in Sec. 13.1, and why this choice led to a nice separation of variables.

13.2.2 Energy conservation

For a closed system, the total energy must be conserved. This means that the Lagrangian does not have an explicit dependence on time for a closed system (as is the case, e.g., for the Lagrangian considered in Section 13.1). And, here, we use the term "energy" to mean the Hamiltonian (see Eqs. 10.13 and 10.14). Note that the total Hamiltonian is the sum of that of the motion of the center of mass and that of the internal motions. For example, for the Lagrangian given by Eqs. 13.10, 13.11, and

[2] This is the so-called particle exchange symmetry.

13.12, the Hamiltonian is given by

$$\begin{aligned} H &= H_{cm} + H_{int}, &(13.14)\\ H_{cm} &= \vec{p}_{\vec{R}} \cdot \dot{\vec{R}} - L_{cm} = L_{cm}, &(13.15)\\ H_{int} &= \vec{p}_{\vec{r}} \cdot \dot{\vec{r}} - L_{int} = \frac{1}{2}\mu\dot{\vec{r}}^2 + U(\vec{r}). &(13.16)\end{aligned}$$

Here, Eq. 13.13 for the total momentum ($\vec{p}_{\vec{R}}$) and

$$\vec{p}_{\vec{r}} = \frac{\partial L}{\partial \dot{\vec{r}}} = \mu\dot{\vec{r}} \tag{13.17}$$

have been used.

13.2.3 Angular momentum conservation

For a closed system, the total angular momentum must be conserved due to the isotropy of space. What would this mean for the Lagrangian? This means that the Lagrangian must be rotationally invariant: when each position vector $\vec{r}_i$ is rotated by an angular displacement $\delta\vec{\phi}$, the Lagrangian must remain invariant.

Let us consider a closed system consisting of two particles interacting by a potential $U(\vec{r}_1 - \vec{r}_2)$. While the homogeneity of space and the homogeneity of time are taken into account by this potential, the isotropy of space is not. For the isotropy of space, it is clear that U cannot depend on the orientation of the vector $\vec{r} = \vec{r}_1 - \vec{r}_2$. So, it can depend only on $r \equiv |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$.

Therefore, we conclude that

$$U(\vec{r}_1 - \vec{r}_2) = U(r). \qquad r = |\vec{r}_1 - \vec{r}_2|, \quad U(r) \text{ is a } \textbf{central potential.} \tag{13.18}$$

By choice, here we are limiting ourselves to the case of a position dependent potential only. In general, however, one finds that the potential function can depend on the velocity as well (as would be the case if charged particles interact via Lorentz force). No matter what the form of the potential is, the potential function must be rotationally invariant. In general, then, the potential energy can be dependent on the velocity as well, and the rotational invariance means that the potential energy be a scalar combination of the velocity vectors and the position vectors involved.

13.3 Two body problem with central force

Central force means that the two-body potential energy, U, belongs in a special simple class: U depends *only* on $r = |\vec{r}_1 - \vec{r}_2|$:

$$U(\vec{r}_1, \vec{r}_2) = U(r). \quad (13.19)$$

As explained in the previous section, this form of interaction energy is a consequence of the basic symmetries of space and time if the interaction energy depends only on position vectors.

With the above potential, the force is only along the radial direction, and there is no angular dependence in the force.

$$\vec{F} = -\hat{r}\,\frac{dU}{dr}. \quad (13.20)$$

From the discussion of the previous section, it should not come as a surprise at all that many fundamental forces of Nature are of this kind: Newton's gravitational law force, Coulomb force, screened Coulomb force, and Yukawa "force" (for nucleons).

From now on, we do not consider the motion of the center of mass, which is a trivial constant velocity motion for a free (compound) particle. We consider the internal motion only.

$$L = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(r). \quad (13.21)$$

Let us note some basic things.

1. This Lagrangian is invariant rotationally, as it should be, as discussed above. This means that the angular momentum vector is conserved: $\vec{L} = \vec{r} \times \vec{p}$. As both $\vec{r}$ and $\vec{v} \propto d\vec{r}$ are perpendicular to the constant vector $\vec{L}$, it follows that $\vec{r}$ should remain in a plane. **It is a planar motion! The problem is a two dimensional one, instead of a three dimensional one.**

2. This Lagrangian, with no explicit time dependence, leads to the energy conservation, as it should, as discussed above. The energy is given by

$$E = \frac{1}{2}\mu|\dot{\vec{r}}|^2 + U(r). \quad (13.22)$$

3. This Lagrangian is apparently *not* translationally invariant on $\vec{r} \to \vec{r} + \delta\vec{r}$, if U is not a trivial constant. But, this fact is of little significance from the point of view of the symmetry of the original problem. From that view, a translation operation means $\vec{r}_1$ or $\vec{r}_2$ is translated, not $\vec{r}$. In fact, $\vec{r} = \vec{r}_1 - \vec{r}_2$ remains constant on translating $\vec{r}_1$ and $\vec{r}_2$ by an arbitrary but fixed amount $\delta\vec{r}$. So, this Lagrangian does in fact have the translational symmetry from this more correct symmetry point of view, as it should. And the system is always characterized by the zero total linear momentum, as $\partial L/\partial\dot{\vec{R}} = 0$, or because the system is "riding the center of mass frame." The motion described by $\vec{r}$ is the relative motion of the constituent particles, and has no bearing on the overall translational symmetry or the total linear momentum.

13.3.1 Angular momentum conservation

For any central force, the angular momentum is conserved. As this means that the motion is effectively two dimensional, we can write the Lagrangian as

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{13.23}$$

in the **two dimensional** polar coordinate system (r, θ). The reason why we prefer the polar coordinate system to the Cartesian system is because the rotational invariance is easier to deal with in the polar system.

The canonical momentum for θ is given by

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = \mu r^2\dot{\theta}. \tag{13.24}$$

This is, as we encountered a similar form a few times by now, the angular momentum associated with the θ rotation. This must be conserved, as L is rotationally invariant on $\theta \to \theta + \delta\theta$ for any constant $\delta\theta$. The Lagrangian EOM verifies that $(\partial L/\partial\theta = 0 = d(p_\theta)/dt)$. So, we have an **"integral of motion"** (cf. Lecture 5).

$$l \equiv p_\theta = \mu r^2\dot{\theta} \tag{13.25}$$

where l is a constant.

Notice that $r^2 d\theta = |\vec{r} \times d\vec{r}|$ $(\because d\vec{r} = dr\hat{r} + rd\theta\hat{\theta}) = |\vec{r} \times (\vec{r} + d\vec{r})|$ $(\because \vec{r} \times \vec{r} = 0)$ is twice the area dA swept by the vector $\vec{r}$ during time dt, as $\vec{r}$ becomes $\vec{r} + d\vec{r}$. (Elaborating this last sentence with a diagram may be very helpful, if this is the first time that you see the angular momentum associated with the areal velocity. That is left for your optional extra work.) Thus,

$$l = 2\mu\frac{dA}{dt}. \tag{13.26}$$

Namely, the angular momentum conservation means a constant areal velocity. This is the general form of **Kepler's second law**, which was discovered for the special form of $U(r) \propto -1/r$.

13.3.2 Energy conservation and the solution

The Hamiltonian is the next integral of motion. Recall that $H = \sum_i p_i \dot{q}_i - L$.

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \tag{13.27}$$

where $p_r = \partial L/\partial \dot{r} = \mu \dot{r}$ has been used as well as the above result for p_θ. This is the energy ($E = K + U$). By using l, we can put it in this form

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r). \tag{13.28}$$

This form nicely fits the general form of the energy that motivates the definition of an effective potential (LN 11). Going further, one can give a physical interpretation of the effective potential as that potential a particle experiences in the reference frame that rotates with the particle. In such a frame the motion of the particle is governed by the effective potential,

$$U_{eff} = \frac{l^2}{2\mu r^2} + U(r). \tag{13.29}$$

The first term in the effective potential is the **centrifugal energy**, which prefers the particle to be at large r. (We have encountered this centrifugal energy term once before in lecture 11.) The solution for this motion is then, if we do as in LN 5,

$$t = \pm \int_{r_0}^{r} dr' \frac{1}{\sqrt{\frac{2}{\mu}[E - U(r')] - \frac{l^2}{\mu^2 r'^2}}}. \tag{13.30}$$

Using $d\theta = l dt/(\mu r^2)$, then

$$\theta - \theta_0 = \pm \int_{r_0}^{r} dr' \frac{l}{r'^2\sqrt{2\mu[E - U(r')] - l^2/r'^2}}. \tag{13.31}$$

Let us analyze the potential energy. Note that the centrifugal energy is a **repulsive energy**, in the sense that the force from it is $l^2/(\mu r^3)$, positive at any r value. Possible motions are determined by the nature of $U(r)$, of course. These are interesting physical cases that we will consider. First, we consider an attractive $U(r)$: $-dU/dr < 0$. Second, $U(r)$ should not be too attractive as $r \to 0$. If $U(r \to 0)$ is too attractive, then all motions will end up at a collision of two bodies at $r = 0$. So, let us restrict to the case when $|U(r \to 0)| <$ the centrifugal term. Third, let us consider the case when $|U(r \to \infty)| >$ the centrifugal term.

These conditions are satisfied by potential energies such as $U = -k/r, kr^2/2$ or kr. The first two are, the Kepler/Newton problem and the Hooke's law problem, respectively, of course. When these conditions are satisfied, there must be a minimum of U_{eff} at some r, since U_{eff} is attractive as $r \to \infty$ (due to $U(r)$), and is repulsive as $r \to 0$ (due to the centrifugal term).

For this class of potential energies, the following lists some typical motions possible. We define turning points as those r values that satisfy $E = U_{eff}$ (cf. Lecture 5).

1. If there is one turning point, and E is equal to the minimum of U_{eff}, then a circular motion ($r =$ constant, $l = \mu r^2\dot{\theta} =$ constant) occurs.
2. If there are two turning points, r_{min} and r_{max}, and neither of them are equilibrium points for $U_{eff}(r)$, then a bound motion occurs. This is an oscillatory motion in r, accompanied by a non-uniform but unidirectional rotation in θ.
3. If there is only one turning point, but E is greater than the minimum of U_{eff}, then it means an unbound motion. The particle will disappear or "escape" to the infinity eventually.

In case 2, the motion can be thought of as two periodic motions in r and θ. These two periods do not need to be commensurate to each other. When the two periods τ_r and τ_θ are commensurate, though, i.e., when $n\tau_r = m\tau_\theta$ for some non-zero integers n, m, then we have the case of a closed orbit, characterized by a finite period of the overall 2D motion. It can be shown ("Bertrand's theorem") that the only power law central forces that can give such a closed orbit motion for $E > U_{eff,min}$ are the $-kr$ force (Hooke's law in 3D) and the $-k/r^2$ force (Newton's law of gravity or Coulomb force for opposite charges).

13.4 Hooke's law problem and the Kepler problem

Let us consider the two most important cases of the central force problem: $U(r) = kr^2/2$ (Hooke's law) and $U(r) = -k/r$ (Kepler problem). As we learned above, we have an effectively 2D problem with a conserved angular momentum. The nature of the motion in the Hooke's law case is easy. It is a 2D SHO. It can be shown that in this case the orbit is a circle, an ellipse, or a line, all with the period $\tau = 2\pi\sqrt{m/k}$ just like the associated 1D SHO that we know and love (the actual work is left for your optional extra work).

In the remainder of this lecture, we discuss the Kepler problem.

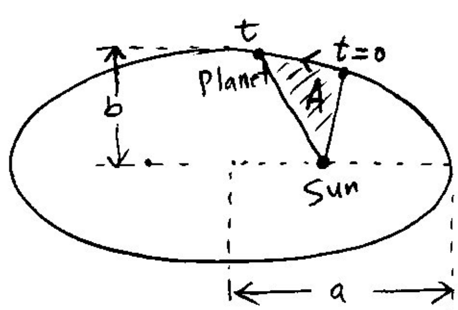
13.5 Kepler's laws

These laws apply to the motion of planets in our solar system. The diagram below is quite **exaggerated** in terms of the difference between a and b. As a matter of fact, most orbits of our planets are close to circles ($a \approx b$).

1. A planet's orbit is an ellipse with the Sun at one of the two focus points. See the diagram, which quite exaggerates the elliptical nature of the orbit. Most orbits are close to circles.

2. The areal velocity dA/dt is constant. See the diagram. The initial point ($t = 0$) can be arbitrarily chosen. A is the cumulative area swept by the position vector of the planet.

3. $\tau^2 \propto a^3$, where τ is the period of the motion, and a is the semi-major axis of the ellipse. a can be replaced by any linear dimension of the ellipse, such as the semi-minor axis b or the mean radius.



13.6 Kepler problem

These observational laws of Kepler can be proven if we use Newton's law of gravity for which

$$U(r) = -\frac{k}{r} \tag{13.32}$$

where $k = GM_S m_p$, M_s is the mass of the Sun, and m_p is the mass of a planet. Note that the reduced mass, relevant for the kinetic energy term of the relative motion is given by $\mu = m_p M_S/(m_p + M_S) \approx m_p$ in the zero-th order as $M_S \gg m_p$. However, we will keep using μ, below, as this problem can describe any two body problem of celestial bodies. For instance, for a binary star consisting of two rotating equally massive stars, the reduced mass will be half of the individual mass.

The solution for the orbit can be found from the θ equation (the last equation of page 8 with $\theta_0 \equiv 0$ and integrating after a change of variable $u = 1/r$: see the derivation below)

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta, \tag{13.33}$$

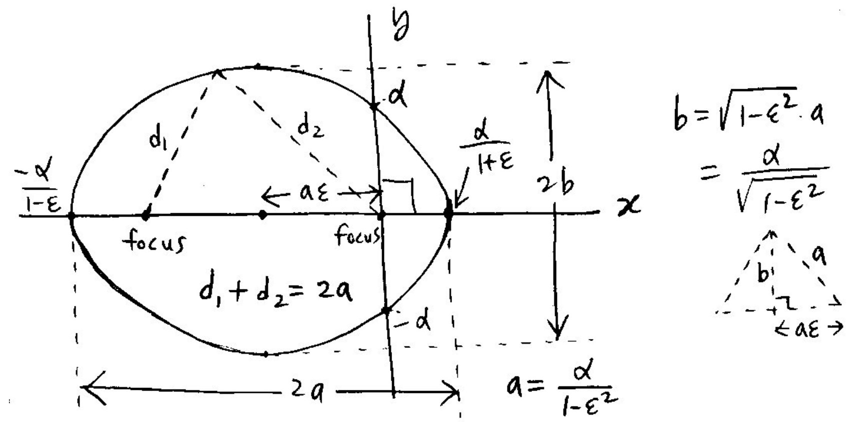
$$\alpha = \frac{l^2}{\mu k}, \tag{13.34}$$

$$\varepsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}}. \tag{13.35}$$

The shape of the orbit is the so-called "conic section" and it depends on the **eccentricity** ε. 2α is called the **latus rectum**.

Orbit	Eccentricity (ε)	Energy (E)	Note
Circle	0	$E = U_{eff,min}$	The easiest, the most important!
Ellipse	$0 < \varepsilon < 1$	$U_{eff,min} < E < 0$	
Parabola	1	$E = 0$	Escape condition, open orbit
Hyperbola	$\varepsilon > 1$	$E > 0$	Open orbit

Here is a summary of the geometry for the elliptical orbit and the circular orbit (**a special case when** $\varepsilon = 0$ **and** $a = b$):



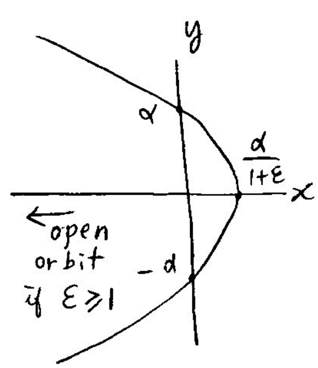
This proves Kepler's first law. The **semi-major axis** a and the **semi-minor axis** b are given by

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|}, \tag{13.36}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}. \tag{13.37}$$

In the diagram, the fact that $a = \alpha/(1-\varepsilon^2)$ follows from the fact that $2a = \alpha/(1+\varepsilon) + \alpha/(1-\varepsilon)$. The fact the center of the ellipse lies at $x = -a\varepsilon$ follows from the fact that that x value must be the mean of $\alpha/(1+\varepsilon)$ and $-\alpha/(1-\varepsilon)$ (min/max of r, or apsides, corresponding to $\theta = 0$ and π). Then, $b = \alpha/\sqrt{1-\varepsilon^2}$ follows from the right triangle shown on the right side of the above diagram, considering the case $d_1 = d_2 = a$.

Before we discuss more about the elliptical orbits, here is a rough diagram for the parabolic or elliptical orbit. It can be considered as an elliptical orbit that becomes too large and eventually open up at the $x \to -\infty$ end as $\varepsilon \to 1$ (parabola) and greater (hyperbola).



13.6.1 Mathematical notes

In the following pages, some mathematical matters are discussed. Despite the fact that I have a label in the next page as "just math," note the fact that without these mathematical steps we would not be able to make progress in science, and so you must make sure that you are capable of following and reproducing these mathematical matters, especially if you aspire to become a theorist. When I say "just math", that is reference to what is required at minimum in *this* course.

Here is a question for your optional extra work. Use the same technique to determine the complete list of orbital shapes when $U(r)$ is the Hooke's law kind ($U(r) = \frac{1}{2}kr^2$).

Optional Reading
"Just Math"

Derivation of

$$\theta = \int_{r_0}^{r} dr' \cdot \frac{\ell}{r'^2 \sqrt{2\mu\left(E - U(r')\right) - \frac{\ell^2}{r'^2}}} \Rightarrow \boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta}$$

$U(r') \to -\frac{k}{r'}$

$\alpha = \frac{\ell^2}{\mu k}$, $\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$

$\frac{1}{r'} = u$, $du = -\frac{dr'}{r'^2}$

$$\theta = -\int_{u_0}^{u} du \cdot \frac{\ell}{\sqrt{2\mu(E + ku) - \ell^2 u^2}}$$

$$\to \sqrt{2\mu E + \frac{\mu^2 k^2}{\ell^2} - \ell^2\left(u - \frac{\mu k}{\ell^2}\right)^2}$$

$$= \ell\sqrt{\frac{2\mu E}{\ell^2} + \frac{\mu^2 k^2}{\ell^4} - \left(u - \frac{\mu k}{\ell^2}\right)^2} \qquad \frac{1}{\alpha} \equiv \frac{\mu k}{\ell^2}$$

$$= \ell\sqrt{\left(\varepsilon^2\right)\frac{1}{\alpha^2} - \left(u - \frac{1}{\alpha}\right)^2} \qquad \varepsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu \ell^2}}$$

$$\theta = -\int_{u_0}^{u} du \cdot \frac{1}{\sqrt{\frac{\varepsilon^2}{\alpha^2} - \left(u - \frac{1}{\alpha}\right)^2}}$$

$$u - \frac{1}{\alpha} = \frac{\varepsilon}{\alpha} \times \cos\xi$$

$$\theta = \cos^{-1}\xi \Rightarrow \xi = \theta$$

$$\frac{1}{r} - \frac{1}{\alpha} = \frac{\varepsilon}{\alpha}\cos\theta \Rightarrow \boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta}$$

The actual mathematical "proof" that the above equation $\frac{\alpha}{r} = 1 + \varepsilon \cos\theta$ really leads to all the conic section curves listed above in page 11, depending on the value of the eccentricity, is given now. Here, by proof, I mean rendering the above polar coordinate equation to the "standard" Cartesian coordinate equation. Note that this equation can be written as $\alpha = r + r\varepsilon\cos\theta$ or $\alpha - \varepsilon x = r$. Squaring both sides, we get $(\alpha - \varepsilon x)^2 = x^2 + y^2$. It is immediately obvious that $\varepsilon = 1$ will render this equation to $\alpha^2 - 2\alpha\varepsilon x = y^2$, which represents a parabola. It is also trivial to see that $\varepsilon = 0$ means $r = \alpha$ (circle). It takes simple algebra to prove that $0 \le \varepsilon < 1$ will render the above equation to the form

$$1 = \frac{\left(x + \frac{\alpha\varepsilon}{1-\varepsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2}. \tag{13.38}$$

Here, a and b are as given above. This fits the general equation of an ellipse in the Cartesian coordinate system $(x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1$. The circle can be considered as a special ellipse, with $\varepsilon = 0$ leading to $a = b$. For $\varepsilon > 1$, we get

$$1 = \frac{\left(x - \frac{\alpha\varepsilon}{\varepsilon^2-1}\right)^2}{\left(\frac{\alpha}{\varepsilon^2-1}\right)^2} - \frac{y^2}{\left(\frac{\alpha^2}{\varepsilon^2-1}\right)}. \tag{13.39}$$

This fits the general equation of a hyperbola in the Cartesian coordinate system $(x - x_0)^2/a^2 - (y - y_0)^2/b^2 = 1$.