Notes for Lecture 12

Gravity

In the previous lecture note, we wrote down Newton's law of gravity in several ways. Here in this note, we explore three examples. The first example is a simple integration problem, while in the other two we make use of the Gauss law.

Here is a question for extra work for motivated students. Consider Earth as a sphere with uniform density. Drill a straight hole through it, passing through the center of the Earth, and drop a mass into that hole. (a) What kind of motion will that mass go through, if we ignore friction. (b) How long will it take to come back to the original drop-off point? As far as figuring out the force field is concerned, this problem is an easier version of Example 12.3, presented below. Now, let us add a twist to this problem. Supposed that the drilled path does *not* go through the center (and so the path will be a shorter one now). Answer questions (a) and (b) for this new straight path, again assuming no friction (however, some normal force *is* required). Let us assume that the Earth is not spinning (although for the first part, the spinning would not matter if the spin axis coincides with the drilled path).

Note for the first example: Here, the problem is considered for $\vec{r}$ in the xy plane. In particular, we set $y = 0$, and evaluate the gravitational potential Φ as a function of x. When done with the calculation, we can replace x with $\rho = \sqrt{x^2 + y^2}$, thanks to the cylindrical symmetry. So, the solution given $\Phi(x, y, z = 0) = -\frac{MG}{2\pi} \int_0^{2\pi} d\phi \, \frac{1}{\sqrt{a^2 - 2ax\cos\phi + x^2}}$ can be rewritten as

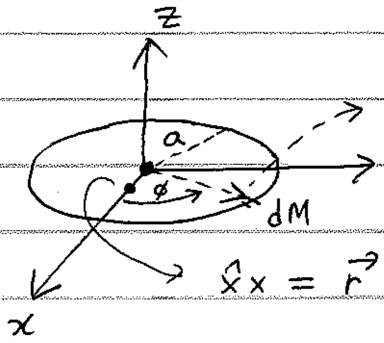
$$\Phi(x, y, z = 0) \;=\; -\frac{MG}{2\pi} \int_0^{2\pi} d\phi \, \frac{1}{\sqrt{a^2 - 2a\rho\cos\phi + \rho^2}}, \qquad \rho = \sqrt{x^2 + y^2}. \tag{12.1}$$

For your optional extra work, you can prove that the procedure $x \to \rho$ is a valid one by a more explicit (but very simple) calculation. Also, it is left for your extra work to write down $\Phi(x, y, z)$ for any z, and prove that the origin is a *saddle point*.

Example 12.1

Gravity example

- Consider $\vec{r}$ in the xy plane only.



$$\vec{r}_M = a\cos\phi\,\hat{x} + a\sin\phi\,\hat{y}$$

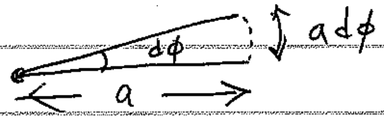
$$\Phi = -\int dM \frac{G}{|\vec{r}-\vec{r}_M|}$$

$$\frac{dM}{a\,d\phi} = \frac{M}{2\pi a} = \text{mass density (linear)}$$

total mass

circumference

$$dM = M\cdot\frac{d\phi}{2\pi}$$



$$|\vec{r}-\vec{r}_M| = \sqrt{(a\cos\phi - x)^2 + a^2\sin^2\phi}$$
$$= \sqrt{a^2 - 2aC_\phi x + x^2}$$

$$\Phi = -\frac{MG}{2\pi}\cdot\int d\phi\cdot\frac{1}{\sqrt{a^2-2aC_\phi x+x^2}}$$

If $|x/a| \ll 1$,

$$\frac{1}{\sqrt{a^2-2aC_\phi x+x^2}} \simeq \frac{1}{a}\cdot\left(1-2C_\phi\frac{x}{a}+\left(\frac{x}{a}\right)^2\right)^{-\frac{1}{2}}$$

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot 4$$

$$\simeq \frac{1}{a}\cdot\left[1+\frac{1}{4}C_\phi\frac{x}{a}-\frac{1}{2}\left(\frac{x}{a}\right)^2+\frac{3}{2}C_\phi^2\left(\frac{x}{a}\right)^2\right]$$

Integrating,

$$\Phi \simeq -\frac{MG}{a}\left(1+\frac{1}{4}\left(\frac{x}{a}\right)^2\right)$$

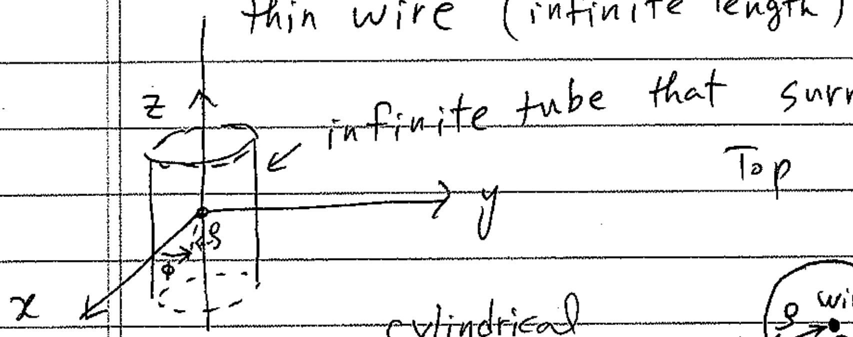
One can change $x \to \rho = \sqrt{x^2+y^2}$ considering the ~~ey~~ symmetry of the prob.

$$\Phi \simeq -\frac{MG}{a}\left(1+\frac{1}{4}\left(\frac{\rho}{a}\right)^2\right) \Leftarrow \left(\begin{matrix}z=0\\\text{case}\\\text{only}\end{matrix}\right)$$

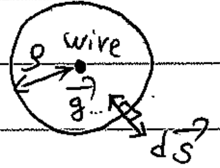
The origin is an unstable equilibrium point. $\nwarrow$ $\left(\frac{\rho}{a}\right) \ll 1$

Example 12.2

Gravity example #2 Gauss law



Top view



(ρ, ϕ, z) ← cylindrical coordinates

(ρ = radius of tube)

Considering a tube (infinite length) of radius

$$\rho = \sqrt{x^2+y^2}$$

and using Gauss law for it

$$\int_{\text{tube}} \vec\nabla\cdot\vec g \, dv = \int_{\text{tube surface}} \vec g \cdot d\vec s = - g \overset{\substack{\text{length of tube or wire } L\to\infty \\ \shortparallel}}{\int dz} \int_0^{2\pi} d\phi \, \rho$$

↑ the magnitude of $\vec g$ constant on the tube due to the cylindrical symmetry

$$= - g L 2\pi \rho$$

$$= -4\pi G M$$

M: total mass of the wire (→∞)

L: length of the wire (→∞)

$A \equiv M/L$ = linear $\overset{\text{mass}}{\vee}$ density of the wire (finite)
= mass per unit length

$$g = 2GA/\rho$$

$$\vec g = -2GA\,\hat\rho/\rho$$

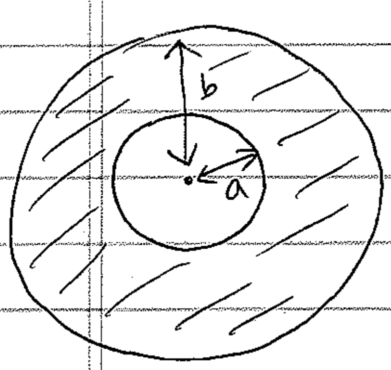
− sign means $\vec g$ towards the origin

$$\Phi = +2GA \ln \rho$$

Example 12.3

Spherically symmetric problem.

Gauss law problem!



Total mass M

mass between $a<r<b$ only

mass "shell"

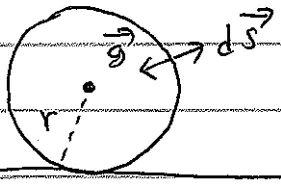
Due to the spherical symm.

$g = |\vec{g}|$, Φ depend only on r

not on θ, ϕ

(r, θ, ϕ) --- spherical coord sys.

Gauss law at any r



$-g \cdot 4\pi r^2 =$ mass inside radius r $\times (-4\pi G)$

① If $r<a$, $g=0$ $\because$ no mass inside

② If $r>b$, $g = \frac{GM}{r^2}$ $\quad \vec{g} = -\frac{GM}{r^2}\hat{r}$

Just as though there is a point mass at the origin!

$\Rightarrow$ Generally true for any spherical mass distribution, if the field is measured completely outside the mass distribution.

③ If $a<r<b$, mass inside $r = \frac{r^3-a^3}{b^3-a^3} \cdot M$

$$\vec{g} = -\frac{GM}{r^2} \cdot \left(\frac{r^3-a^3}{b^3-a^3} \right) \cdot \hat{r}$$

Note that $\vec{g}(r=a) = 0$ --- agrees with ①

$\vec{g}(r=b) = -\frac{GM}{b^2}\hat{r}$ ... agrees with ②

$\therefore$ $\vec{g}$ is continuous everywhere.

why is $\vec{g}$ continuous ?

because mass density is only moderately discontinuous (like a step function)

(→ $\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$ (ρ : mass density)

→ $\vec{g}$ is an integral of ρ

If ρ is only moderately discontinuous (piece-wise conti like a step ftn)

then $\vec{g}$ will be continuous.)

Next, get Φ from $\vec{g} = -\vec{\nabla}\Phi$

① If $r<a$, $\Phi = \text{const.} = C$

② If $r>b$, $\Phi = -\frac{GM}{r} + \underbrace{\text{const}}_{\text{choose to be } 0} \equiv -\frac{GM}{r}$

Just fixing the over-all offset of Φ. ←-- by requiring $\Phi(r\to\infty)=0$

(One way to do it)

③ If $a<r<b$, $\Phi = \frac{GM}{b^3-a^3}\left(\frac{r^2}{2}+\frac{a^3}{r}\right) + A$ ← const.

Note $\vec{g}$ continuous → Φ must be continuous.

(actually, differentiable !)

Determine A, and C by continuity.

② ... $\Phi(r\to b) = -\frac{GM}{b}$ ③ ... $\Phi(r\to b) = \frac{GM}{b^3-a^3}\left(\frac{b^2}{2}+\frac{a^3}{b}\right)+A$

Solving for A we get $A = -\frac{3}{2}\cdot\frac{GM}{b^3-a^3}\cdot b^2$

$\therefore$ ③ ... $\Phi = \frac{GM}{b^3-a^3}\left(\frac{r^2}{2}+\frac{a^3}{r}-\frac{3}{2}b^2\right)$

as $r\to a$, this one goes to $\Phi(r\to a) = \frac{3}{2}\cdot\frac{GM}{b^3-a^3}\cdot(a^2-b^2)$

This is the value for ① $r<a$ by continuity.