Notes for Lecture 11

Effective potential, Constraint, Gravity

In previous lectures, we discussed the action principle and the subsequent equations of motion at length: Lagrange equation of motion (where L is the main quantity) or the canonical equation of motion (where H is the main quantity). Newton's equation of motion, Lagrange equation of motion, and the canonical equation of motion are all a valid and equivalent way of expressing a mechanical problem, and the choice can be made depending on your preference or the nature of the question.

It is important to note, however, that new ways of studying mechanics starting from the least action principle naturally place a great emphasis on the role of symmetry, a very important aspect of modern physics.

Here, we continue our discussions on action based principles, and point out some important practical quantities for the Hamiltonian (H) based formalism and the Lagrangian (L) based formalism. Then, we will start the application of our new formalisms to the Kepler problem.

11.1 Effective potential

Let us consider a problem for which the Lagrangian is not explicitly dependent on time. So, the Hamiltonian is conserved. Suppose that the Hamiltonian can be written as $\frac{1}{2}A(q)\dot{q}^2 + f(q)$ where $A(q) \geq 0$ is a function of q, but not a function of $\dot{q}$, and $f(q)$ is a function of q and other parameters of the problem, including any other conserved quantities (e.g., angular momentum). In this case, it is very useful for us to consider

$f(q)$ as our *effective potential* and write

$$H = \frac{1}{2}A(q)\dot{q}^2 + U_{eff}(q). \tag{11.1}$$

Here are some points to consider in this situation, which occurs quite frequently.

1. Just because the Hamiltonian can be written in terms of only one generalized coordinate does not mean that the Lagrangian contains only one generalized coordinate. The actual motion may very well involve multiple degrees of freedom, and thus multiple generalized coordinates. See the example below. Still, the Hamiltonian can end up depending only on one generalized coordinate and its time derivative if there are other constants of motion, i.e., conserved quantities.

2. What the above expression for a conserved H means is that the motion can be considered effectively one dimensional, and for this effective one dimensional motion in q, we can use our results in lecture 5. In particular, the solution $q(t)$, or rather $t(q)$, can be readily found as follows.

$$t = \pm \int_{q_0}^{q} dx' \sqrt{\frac{A(x')}{2(H - U_{eff}(x'))}}. \tag{11.2}$$

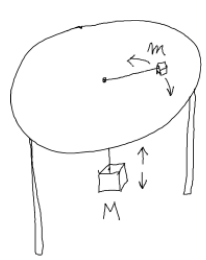
3. $A(q)$ can often be regarded as an effective mass or an effective rotational inertia.

4. The first term in the above equation for H can be regarded as the effective kinetic energy. Just because one can write (as above)

$$H = K_{eff} + U_{eff}$$

where $K_{eff} \equiv \frac{1}{2}A(q)\dot{q}^2$, does *not not* mean, in general, that $L = K_{eff} - U_{eff}$. It may happen to be in some cases, but it is not in many other cases.

11.1.1 A simple example



In this example, we consider a mass m connected to another mass M below a table, by a taut string. We assuming that there is no friction.

The small mass m has two degrees of freedom, for its movement on the table. We can call them r and θ, using the polar coordinate system. There is no potential energy involved with r and θ. The kinetic energy is given by $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$ in the Cartesian coordinate system. You can convert it to the polar expression, $\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$, using $x = r\cos\theta$ and $y = r\sin\theta$ (please do work out this conversion yourself, if you have never done it before, or if your memory is vague).

Now, let us consider mass M. This mass has both the kinetic energy ($\frac{1}{2}M\dot{r}^2$) and the potential energy (Mgr). The potential energy is given by $Mgz = -Mg(l-r)$, if the z axis is measured up from the center of the table, and this potential energy is equal to Mgr up to a constant offset.

Thus, the Lagrangian is given by

$$L = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - Mgr. \tag{11.3}$$

Note that $\frac{\partial L}{\partial t} = 0$ and so the Hamiltonian is conserved. Also, note that $\frac{\partial L}{\partial \theta} = 0$ and so $p_\theta = \frac{\partial L}{\partial \dot{\theta}}$ is conserved. This canonical conjugate momentum is, of course, the angular momentum L_z, and this arises from the rotational symmetry of this problem: the Lagrangian remains invariant if both masses are rotated by any fixed but arbitrary amount around the z axis—the vertical axis going through the center of the table. Here are canonical conjugate momenta and the Hamiltonian function.

$$p_r = \frac{\partial L}{\partial \dot{r}} = (M+m)\dot{r}, \qquad \textit{not}\text{ conserved} \tag{11.4}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad L_z\text{: conserved} \tag{11.5}$$

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = \frac{p_r^2}{2(M+m)} + \frac{p_\theta^2}{2mr^2} + Mgr. \qquad H\text{: conserved} \tag{11.6}$$

In this problem, there are two degrees of freedom, r and θ. On the other hand, the conserved Hamiltonian is a function of $\dot{r}$ and r, thanks to the conserved angular momentum p_θ.

$$H = \frac{1}{2}(M+m)\dot{r}^2 + U_{eff}(r) = \text{constant}, \tag{11.7}$$

$$U_{eff}(r) \equiv \frac{p_\theta^2}{2mr^2} + Mgr. \tag{11.8}$$

Note that the effective potential defined in this way has a minimum. To see this, first note that $r \geq 0$. As $r \to 0$, $U_{eff}(r) \to \frac{p_\theta^2}{2mr^2} \to \infty$, and, as $r \to \infty$, $U_{eff}(r) \to Mgr \to \infty$.

This means that $U_{eff}(r)$ should have at least one minimum. Taking the derivative, we see that $\frac{dU_{eff}}{dr} = Mg - \frac{p_\theta^2}{mr^3} = 0$ gives only one physical solution ($r \geq 0$): $r_{eq} = \sqrt[3]{\frac{p_\theta^2}{mMg}}$. So, it follows that r_{eq} corresponds to a minimum of the effective potential, and we can also conclude that the effective potential has no other extrema. For now, let us pretend that the r motion only. At $r = r_{eq}$, we have a stable equilibrium, and so we have a stationary value of radius $r = r_{eq}$. If the energy is slightly higher than the minimum value of U_{eff}, then

$$U_{eff}(r) \approx U_{eff}(r_{eq}) + \frac{1}{2}\frac{3p_\theta^2}{mr_{eq}^4}(r - r_{eq})^2, \tag{11.9}$$

$$\omega_r^2 = \frac{3p_\theta^2}{m(m+M)r_{eq}^4} = \frac{3m^{1/3}(Mg)^{4/3}}{(m+M)p_\theta^{2/3}}. \tag{11.10}$$

Here, ω_r is the frequency for the radial simple harmonic motion near the bottom of U_{eff}.

In all this, one must not forget that the angular motion occurs at the same time, and so describing the radial motion addresses only one part of the motion. The relation between the angular motion and the radial motion is governed by the angular momentum conservation $p_\theta = mr^2\dot{\theta} =$ constant. One may wonder what kind of shape of orbit one obtains. It turns out that in the current case the orbit is, in general, an open orbit, when the energy is not equal to the minimum of the effective potential.

11.2 Lagrangian mechanics with constraints

Why should we consider Lagrangian mechanics with constraints? The reason can be understood by considering certain example problems. For instance, we may be considering a problem such as a particle sliding down a circular shape and asking "when does the particle take off from the surface?". This is the problem discussed below. Or, we may be considering a particle starting a loop-the-loop roller coaster type movement and asking "what is the minimum initial speed necessary for the particle to complete the circular motion?". As a third example, let us say that you are given a cylindrical object with total mass M and radius R. You are told that the inside this hollow cylinder there is a steel ball of known radius r inside the cylinder. Indeed, you can hear it. You are then asked to figure out the mass of the steel ball without destroying the cylinder object. One way to do so is by rolling it by a small amount, inducing a simple harmonic motion (assuming rolling without slipping between the steel ball and the cylinder, as well as between the cylinder and the table), and measuring the frequency of the simple harmonic motion. Then, suppose you or someone asks, how much friction is there between the steel ball and the inner surface of the cylinder or between the table and the outer surface of the cylinder?

The first two examples are common problems considered in elementary Newtonian mechanics courses. (And, they can be made a bit complicated by making the particle roll, without slipping, instead of sliding.) The last example is somewhat complicated, while the rolling without slipping is definitely a subject dealt with frequently in such courses.

There is a systematic method to answer the above questions in the Lagrangian formalism. And this method involves the **Lagrange multipliers**. We shall see that a Lagrange multiplier gives information on the **generalized force** (Section 9.3) responsible for the constraint.

Here, we consider an important class of constraints that can be expressed as[1]

$$a_t(\{q_j\},t) + \sum_i a_i(\{q_j\},t)\dot{q}_i \; = \; 0. \qquad (11.11)$$

Equivalently, this constraint can be written as

$$a_t(\{q_j\},t)dt + \sum_i a_i(\{q_j\},t)dq_i \; = \; 0. \qquad (11.12)$$

As the coefficients a_i and a_t are functions of positions and time, there is no guarantee that the above constraints are integrable. So, in general, we say that this constraint is *non-holonomic*. Integrable constraints are called *holonomic*. For example, if Eq. 11.12 can be expressed as a total differential, $dg(\{q_j\},t) = 0$, then the constraint is holonomic.

The above constraint, while generally non-holonomic, admits the use of the Lagrange multiplier. To see this, we must go back all the way to the least action principle. The variation in action δS for M degrees of freedom is given by

$$\delta S \; = \; \int_1^2 dt \sum_{i=1}^{M} \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right\} \delta q_i(t). \qquad (11.13)$$

This equation is the multi-degrees of freedom version of Eq. 9.5, and, by setting $\delta S = 0$, the Lagrange equation of motion Eq. 9.9 is derived, *if* there are no constraints on q_i's, i.e., if all variations $\delta q_i(t)$ are independent.

[1] The most general constraint that one can imagine within a Lagrangian formalism can be written as $F(\{q_i\},\{\dot{q}_i\},t) = 0$, where F is a function, $\{...\}$ refers to the set of M variables for $i = 1, ..., M$, where M is the degrees of freedom. If this constraint is integrable to give a constraint of the form $G(\{q_i\}) = 0$, where G is a function, then we say that the constraint is *holonomic*. For a holonomic constraint, we can simply invert the function G and reduce a number of generalized coordinate. But, not all important constraints are holonomic. The constraint that we consider in Eq. 11.11, or Eq. 11.12, is generally non-holonomic.

Now, we ask, what if there are constraints? Clearly, when there are constraints (of the form Eq. 11.12), not all δq_i's are independent. If there are N constraints, then only $M - N$ generalized coordinates are independent.

Before we make use of the constraint equation, Eq. 11.12, one thing to note is that in the variation considered for the least action principle time is always fixed. We say that the variation $\delta q_i(t)$ is *virtual*, not real. So, for the variation that we consider for the least action principle, the first term in Eq. 11.12 vanishes, and the constraint equation becomes

$$\sum_i a_i \delta q_i = 0, \tag{11.14}$$

where, despite the simplicity of notation employed here, a_i's are to be considered as, in general, functions of generalized coordinates and time.

The Lagrange multiplier technique is motivated by the trivial fact that, if we add a zero to the integrand of Eq. 11.13, then nothing changes. In particular, λ times Eq. 11.14 is still zero for any number λ, which is called the **Lagrange multiplier**. Just like L, λ can be taken as a function of generalized coordinates, generalized velocities, and time. Then, Eq. 11.13 can be rewritten as

$$\delta S = \int_1^2 dt \sum_{i=1}^{M} \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + a_i \lambda \right\} \delta q_i(t) \equiv \int_1^2 dt \sum_i f_i \delta q_i. \tag{11.15}$$

Here, we have assigned the symbol f_i to the expression in curly brackets. Note that we used one constraint and so there is one Lagrange multiplier appearing in this equation. Since λ can be any number, we are free to choose it so that $f_1 = 0$, for example. Doing so, we see that the sum in the integrand becomes $\sum_{i=2}^{i=M} f_i \delta q_i$. Now, this sum is over only $M - 1$ variations, δq_i's with $i \neq 1$, and these variations *are* independent. Therefore, when we apply the least action principle $\delta S = 0$, we get $f_i = 0$ for $i = 2, ..., M$. So, what happened? All f_i's are zeros, thanks to the Lagrange multiplier, and we get the following Lagrange equations of motion with constraint.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda a_i, \qquad i = 1, ..., M. \tag{11.16}$$

This and the constraint equation, $a_t + \sum_i a_i \dot{q}_i = 0$, (Eq. 11.11), provide $M+1$ coupled equations necessary for solving for $M + 1$ unknowns, q_i's and $\lambda(\{q_i\}, \{\dot{q}_i\}, t)$.

If there are multiple constraints, say N constraints,

$$a_{t,j} + \sum_{i=1}^{M} a_{i,j} \dot{q}_i = 0 \qquad j = 1, ..., N \tag{11.17}$$

then, the above equation of motion is generalized to (the proof is left for your exercise)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{N} \lambda_j a_{i,j}, \qquad i = 1, ..., M. \tag{11.18}$$

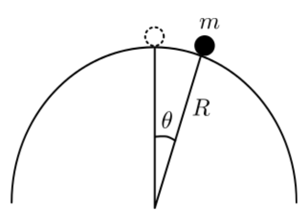
In this form, note that there are N Lagrange multipliers introduced, one per each constraint.

What does the Lagrange multiplier mean physically? It alone does not have a physical meaning, but in combination with the constraint coefficient a_i, λa_i in Eq. 11.16 or $\sum_{j=1}^{N} \lambda_j a_{i,j}$, has the physical meaning of the generalized force of constraint (cf., Section 9.3). This is the force or the torque that is responsible for the constraint.

11.2.1 A simple example

Consider a perfectly smooth hemispherical hill on top of which there is a particle at rest. This particle is in an unstable equilibrium (show it by inspecting its potential energy) and it is soon observed to slide down the circular slope of the hill. The question is whether it will fly off the hill, and if so at what angular position.

For analyzing the motion of this particle, we can use the polar coordinate system, where θ measures the angular displacement from the top of the hill and r is the distance of the particle from the center of the full circle, the top half of which is the shape of the hill.



The constraint here is simply that $r = R$. Therefore, this is a holonomic constraint. We can put this constraint in the form of Eq. 11.12, by taking the differential

$$dr = 0. \tag{11.19}$$

So, we can see that $a_\theta = 0$ and $a_r = 1$. Here, we use the generalized coordinate for the subscript, rather than the index for the generalized coordinate, for clarity.

The Lagrangian of this problem is given by

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\cos\theta. \tag{11.20}$$

The equations of motion, Eq. 11.16, become

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta \;=\; \lambda, \qquad \text{from } \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = a_r\lambda \tag{11.21}$$

$$mr^2\ddot{\theta} \;=\; mgr\sin\theta. \qquad \text{from } \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = a_\theta\lambda \tag{11.22}$$

Here, the second equation is the familiar equation, $r\ddot{\theta} = g\sin\theta$, and the first equation, if we *now* use the constraint $r = R$, becomes

$$\lambda \;=\; -mR\dot{\theta}^2 + mg\cos\theta. \tag{11.23}$$

This is the force that is responsible for keeping $r = R$. What is the physical meaning of it? First, note that $mg\cos\theta$ is the gravitational force in the radial direction. Second, $mR\dot{\theta}^2$ is the centripetal force. The difference between the two is the normal force, which is the meaning of (t). The normal force must exist between two surfaces in contact, and, barring any (natural or glue-caused) adhesive tendency between two surfaces, it must be positive, as is the assumed case here.

So, the angle at which the particle takes off is given by

$$\lambda \;=\; -mR\dot{\theta}^2 + mg\cos\theta \;=\; 0. \tag{11.24}$$

This does not give answer for the angle of take-off, but we can remember that the total energy is conserved in this problem (see the next section for more discussion on this),

$$H \;=\; p_r\dot{r} + p_\theta\dot{\theta} - L \tag{11.25}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta, \qquad \text{general} \tag{11.26}$$

$$= \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta, \qquad \text{with constraint } r = R \tag{11.27}$$

$$= mgR \qquad \text{initial energy (at rest at top)} \tag{11.28}$$

Using the above take-off condition $g\cos\theta = R\dot{\theta}^2$, we see that the energy conservation equation can be rewritten as

$$\frac{1}{2}\cos\theta + \cos\theta \;=\; 1. \tag{11.29}$$

The solution to this equation gives the take off angle

$$\theta_{take-off} \;=\; \cos^{-1}\frac{2}{3} \;\approx\; 48.2^\circ. \tag{11.30}$$

Admittedly, doing this problem with the Lagrangian formalism is more complicated than doing it with the good-old Newtonian mechanism. But, the Lagrangian formalism shines for more complicated problems.

11.2.2 Notes on the Lagrangian formalism with constraints

It is important to remember the following points, when solving a Lagrangian problem with constraint(s).

1. When writing down the Lagrangian and the Lagrange equation of motion, do *not* use the constraint. Treat all q_i's as fully time-dependent and variable. In the above example, note that $\dot{r}$ appears in L, as does $\ddot{r}$ in the radial equation of motion.

2. *After* setting up the Lagrange equation of motion, then the constraint equation can be used to simplify the equation of motion.

3. The conservation principles do not work in general if the Lagrangian formalism is with constraints. The Hamiltonian conservation principle may well break down even if L has no explicit time dependence but there is an explicitly time dependent constraint. Also, even if $\frac{\partial L}{\partial q_i} = 0$, the canonical conjugate momentum $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ may not be conserved if the generalized force of constraint for q_i is not zero: this is because of Eq. 11.18 or Eq. 11.16.

4. The generalized force of constraint, given by $\sum_j \lambda_j a_{i,j}$, is a reaction force (or a reaction torque) in general, where the word reaction is used in the Newton's third law sense of action-reaction pair. For instance, one can analyze the problem of a ball rolling down an incline without slipping, in a constant gravitational field. In such case, the force of constraint is a rolling friction or the torque cause by it. You are encouraged to consider this problem yourself and find the friction and the torque caused by it. Also, if you like more challenges, do the example problem that we described in the previous section with rolling without slipping (i.e., two surfaces of contact have no relative velocity) assuming that the particle is a spherical ball (or any other shape with a circular crossection) that rolls down the hill—such problem will now involve two constraints. Or, do the complicated example described at the beginning of Section 11.2—that is also a problem with two constraints.

5. For a simple problem like the example we considered in the previous section, the constraint is holonomic, and the problem does not need a treatment of the Lagrangian-with-constraints formalism. Namely, if the constraint $r = R$ is used in that problem from the beginning, then the Lagrangian will be simply $L = \frac{1}{2}mR^2\dot{\theta}^2 - mgR\cos\theta$, and we have an effectively one dimensional problem. One can go ahead and do mechanics with that Lagrangian. For instance, by doing so, one will discover that H is conserved (and, one can see easily that H continues to be conserved after the particle takes off the hill). However, such

treatment will be valid only up to the θ value at which the particle takes off from the hill. Also, such treatment alone will not inform us about when the take-off occurs and how the constraint force changes during the motion (unless, of course, one sprinkles the problem with some Newtonian mechanics knowledge by inspecting a free-body diagram).

11.3 Effective potential, revisited

In Section 11.1, we discussed the effective potential when there are *conserved quantities*.

Now that we have learned the Lagrangian formalism with constraints, a question arises, *what if the effective potential results from a (holonomic) constraint*, not from a conservation law? In such a case the question is whether the constraint must be used while setting up the Lagrangian, or we must use the Lagrangian with constraints formalism. In such a case, one must consider point 3 of Section 11.2.2 carefully.

The fundamental rule to follow in this case, or in any case, is that the Hamiltonian must be a conserved quantity for the effective potential concept to be useful. Only if this is true, then the effective potential defined in Eq. 11.1 is useful.

Let us consider an example (for your optional extra work). Suppose that there is a circle shaped hard wire with a bead sliding along it without friction. The wire is placed in a vertical plane. It is held at the top point and then it is rotated around the vertical axis (that contains the top point and the bottom point of the wire) at a constant angular speed ω. The constant gravitational field of magnitude g acts downward. What is the appropriate effective potential energy to describe the motion of the bead and how does the nature of the motion change as a function of ω? In such a problem, one can easily see that the description of the bead motion must be reduced to that of a one dimensional motion, as the motion of the bead is three dimensional with two constraints (the bead on the circle, and the circle being rotated at angular speed ω). In this example, you can show that the Hamiltonian is conserved, if the two constraints are used while the Lagrangian is set up (as is often the case; see point 3 of Section 11.2.2) but the Hamiltonian is *not* conserved if it is derived within the Lagrangian plus constraints formalism. So, in this example, the constraints must be used *while* setting up the Lagrangian if the effective Hamiltonian concept is to be used. In other words, the Lagrangian must already have only one degree of freedom in this example, if the effective potential concept is to be useful. As this example shows, *constraints* and *conserved quantities* must be treated quite differently when dealing with the effective potential: the only guiding principle to remember in either case is the Hamiltonian conservation.

11.4 Gravity

11.4.1 Newton's law of gravity

For two bodies, there is an attractive force of the magnitude

$$F = G\frac{Mm}{r^2} \tag{11.31}$$

and the direction which is parallel to the line joining the two bodies. Here, M, m are the masses of the two bodies. r is the distance between them.

11.4.2 Gravitational field

Field is an important modern concept. It does away the "action at a distance," which Newton himself had a hard time believing (and so did Einstein).

Consider a body of mass at M found at some point. Let us conveniently take that point to be the origin. Then, we say that the gravitational field, $\vec{g}$, at position vector $\vec{r}$ due to this mass M is

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \tag{11.32}$$

where $r = |\vec{r}|$, and $\hat{r}$ is the radial unit vector $\hat{r} = \vec{r}/r$. Observe that taking the position of mass M was purely for convenience. In general, we can change $\vec{r} \to \vec{r} - \vec{r}_M$, and $\hat{r} \to (\vec{r} - \vec{r}_M)/|\vec{r} - \vec{r}_M|$, where $\vec{r}_M$ is the position of mass M, and everything is good.

Then, the force that another mass m feels due to M is given by

$$\vec{F}_m = m\vec{g}. \tag{11.33}$$

Two comments. (1) We are defining $\vec{g}$ generally here, not just the Earth gravity near its surface. In general, $\vec{g}$ is position dependent, not constant. (2) The definition of a field is a mathematical triviality, at this level. But, imagine that the field is some sort of a real thing that connects two massive bodies! The concept of the field is a big deal, while the particles we think are responsible for the gravitational field (gravitons) haven't been detected by any human scientific equipment yet (compare this situation with photons which are responsible for the electromagnetic field).

11.4.3 Gravitational potential

Let us look at the mathematics of the field a bit: $\vec{g}(\vec{r}) = -GM\hat{r}/r^2$. This is a conservative field. Which means two things:

$$\vec{\nabla} \times \vec{g} = 0, \tag{11.34}$$

$$\vec{g} = -\vec{\nabla}\Phi(\vec{r}). \tag{11.35}$$

Here, Φ is related to the potential energy U via $U = m\Phi$, where m is the other mass that interacts with mass M. The gravitational potential Φ can be obtained as

$$\Phi(\vec{r}) = -\frac{GM}{r}. \tag{11.36}$$

Indeed, the existence of such a potential proves that the gravity is a conservative force. Or, generally, if the mass M is not at the origin, but at $\vec{r}_M$:

$$\Phi(\vec{r}) = -\frac{GM}{|\vec{r} - \vec{r}_M|}. \tag{11.37}$$

Let us consider a simple fact. If there are multiple bodies, then the total force is obviously the addition of all forces. Each force can be considered as coming from a potential field. It then follows that the potential field is also additive. This is due to the linear operator nature of the $\vec{\nabla}$ operator that connects the potential and the field. So, for a gravitational field that arises from multiple bodies,

$$\Phi(\vec{r}) = -G\sum_i \frac{M_i}{|\vec{r} - \vec{r}_{M_i}|} \tag{11.38}$$

$$\vec{g} = -G\sum_i \frac{M_i}{|\vec{r} - \vec{r}_{M_i}|^3}(\vec{r} - \vec{r}_{M_i}) \tag{11.39}$$

For a continuous distribution of masses, these change to the integral:

$$\Phi(\vec{r}) = -G\int \frac{dM}{|\vec{r} - \vec{r}_M|} \tag{11.40}$$

$$\vec{g} = -G\int dM \frac{\vec{r} - \vec{r}_M}{|\vec{r} - \vec{r}_M|^3} \tag{11.41}$$

11.4.4 Gauss law, Poisson equation

Let us consider a volume integral

$$\int_V dV \vec{\nabla} \cdot \vec{g} = \int_S d\vec{S} \cdot \vec{g}. \tag{11.42}$$

The above equality of the volume integral of a divergence of a vector field and the surface integral of the vector field is called **Gauss's theorem**. It is very important. Here, S is the surface area of the volume V, and $d\vec{S}$ is a small area element vector. The magnitude of $d\vec{S}$ is the area of the area element, and the direction of it is normal to the area element and towards the outside of the volume.

Consider the simplest case first. Assume $\vec{g} = -GM\hat{r}/r^2$, and the volume V is a sphere of a radius R, centered at the origin. The integral is then

$$\begin{aligned}\int_V dV\vec{\nabla}\cdot\vec{g} &= \int_S d\vec{S}\cdot\vec{g} &(11.43)\\ &= -\frac{GM}{R^2}\int_0^{\pi} d\theta\int_0^{2\pi} d\phi\sin\theta\, R^2 &(11.44)\\ &= -GM\int d\Omega &(11.45)\\ &= -4\pi GM &(11.46)\end{aligned}$$

Here, $d\Omega$ is the infinitesimal solid angle[2]

$$d\Omega \stackrel{def}{=} \frac{d\vec{S}\cdot\hat{r}}{r^2} \qquad (11.47)$$

subtended by the area element $d\vec{S}$ at the origin. Using the concept of the solid angle, the above integral immediately generalizes to any volume, which may or may not enclose the origin:

$$\begin{aligned}\int_V dV\vec{\nabla}\cdot\vec{g} &= \int_S d\vec{S}\cdot\vec{g} &(11.48)\\ &= -GM\int d\vec{S}\cdot\hat{r}/r^2 &(11.49)\\ &= -GM\int d\Omega &(11.50)\end{aligned}$$

Note that whenever the volume V encloses the origin, then it is $\int d\Omega = 4\pi$, while if V does not enclose the origin, then $\int d\Omega = 0$.

$$\begin{aligned}\int_V dV\vec{\nabla}\cdot\vec{g} &= -4\pi GM \qquad \text{if } V \text{ encloses the mass } M\text{, the source of } \vec{g}, &(11.51)\\ &= 0 \qquad \text{if it does not,} &(11.52)\end{aligned}$$

[2]This is the general definition of the infinitesimal solid angle for an area element $d\vec{S}$ which is at the position vector $\vec{r}$. Note that it can be positive or negative depending upon whether the normal vector of the area element is pointing away from the origin or towards it. This should not be surprising, given the fact that the "linear" angle also has a sign. In the spherical coordinate, the volume element $dV = r^2\sin\theta\, drd\theta d\phi = r^2drd(\cos\theta)d\phi$. This is **always** equal to $dV = r^2drd\Omega$. This may be used as an alternative definition of $d\Omega$, if you like, and is equivalent to the above definition.

for *any volume* V.

What if the mass M is not placed at the origin? In that case, $\int dV\vec{\nabla}\cdot\ldots = \int dV_M\vec{\nabla}_M\cdot\ldots$ by a mere translation of the coordinate vectors, where the subscript M means the coordinate system whose origin is at $\vec{r}_M$. Therefore, the above result is valid even if M is displaced from the origin. [**Note 1**: This simply means that if we shift the position of the mass and the volume of integration at the same time, the integral should not change at all. This is obvious since we are free to choose the origin at any point that we like. **Note 2**: The key point to remember here is that no matter how we deform the volume it will remain invariant as long as the mass stays inside the volume, if the initial volume enclosed the mass, or the mass stays outside the volume, if the initial volume did not enclose the mass. For instance, suppose we start from a sphere with a mass M at the origin of the sphere. If we shift the sphere a little so that now the mass M is off-center with respect to the sphere, the above integral remains invariant as long as M continues to stay inside the sphere.]

Therefore, the above result then immediately generalizes to the case when there is any distribution of masses, not just one mass.

So, what we have is the generalization of the above result:

$$\int_V dV\vec{\nabla}\cdot\vec{g} = \int_S d\vec{S}\cdot\vec{g} = -4\pi GM. \tag{11.53}$$

V is *any volume*.
M is the total mass enclosed by V, including 0.
$\vec{g}$ is the total gravitational field, due to *all masses* around, not just M.

Since V is any volume, it can be taken to be dV. Then, $M = \rho(\vec{r})dV$, where $\rho(\vec{r})$ is the mass density. Then, we have

$$\vec{\nabla}\cdot\vec{g} = -4\pi G\rho(\vec{r}). \tag{11.54}$$

These two boxes above contain an extremely useful law of physics: the **Gauss's law** for the gravity. Gauss's law is applicable to Newton's law of gravity and Coulomb's law of the electrostatic force.

Gauss's law can be re-written in terms of Φ,

$$\vec{\nabla}^2\Phi = 4\pi G\rho(\vec{r}). \tag{11.55}$$

This is the **Poisson's equation** for the potential Φ. It is a kind of differential equation, like the SHO ODE that we solved is a differential equation, except that in the current case the differential equation involves many variables, as many as the spatial dimension.

For a point mass at $\vec{r}_M$, $\rho(\vec{r}) = M\delta(\vec{r}-\vec{r}_M)$, where $\delta(\vec{r}-\vec{r}_M)$ is the Dirac-delta function in three dimensions[3], note that we know the solution Φ already. It is $-GM/|\vec{r}-\vec{r}_M|$. So, we can write

$$-\vec{\nabla}^2\frac{1}{4\pi|\vec{r}-\vec{r}_M|} = \delta(\vec{r}-\vec{r}_M). \tag{11.56}$$

That is, $-\frac{1}{4\pi|\vec{r}-\vec{r}_M|}$ is the **Green's function** of the Poisson equation.

11.4.5 Meaning of the gradient

Suppose you make a plot of equipotential surfaces, i.e. a collection of surfaces, which satisfy Φ = constant. In which direction does the force field, $\vec{g}$ point? Due to the nature of the gradient, $\vec{g}$ always points perpendicular to the equipotential surface. This does not uniquely determine the direction of $\vec{g}$. It could point along the direction in which the potential increases, or the direction in which the potential decreases. As $\vec{g}$ is the negative gradient of Φ, $\vec{g}$ points towards the direction in which Φ decreases. Also, note that the force field is greater in magnitude where the equipotential surfaces are dense.

[3]Some authors choose to write this as $\delta^3(\vec{r}-\vec{r}_M)$.