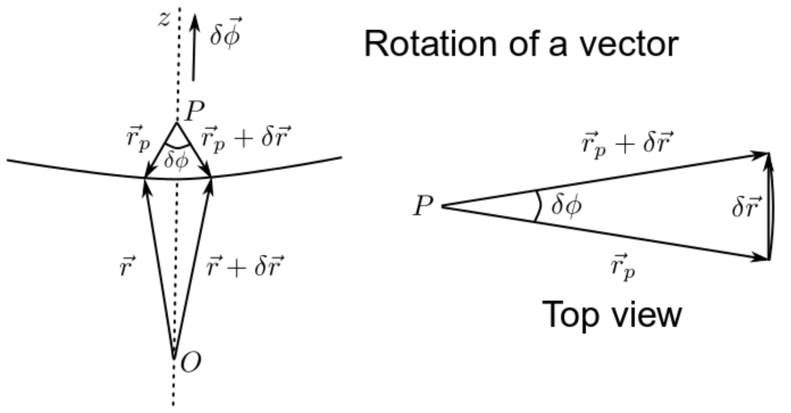
Notes for Lecture 10

Symmetry–Conservation, Hamiltonian

Let us continue the discussion of symmetry and conservation. This discussion will lead to the concept of Hamiltonian as a generalization of the energy concept.

10.1 Angular momentum conservation

Let us consider rotating an arbitrary position vector $\vec{r}$ by angle $\delta\phi$.



The above diagram shows rotating a position vector $\vec{r}$ around an axis, which is labeled z. The origin of the position vector is taken to be O, which is assumed to be on the axis of rotation. The vector $\vec{r}$ is rotated by a small angle $\delta\phi$ to become $\vec{r} + \delta\vec{r}$.

Note that there is a unique plane perpendicular to the z axis that contains both the end points of $\vec{r}$ and $\vec{r} + \delta r$. This plane can be visualized in the diagram as having the center point P and containing $\vec{r}_p$ and $\vec{r}_p + \delta\vec{r}$. Namely, $\vec{r}_p$ is the projection of $\vec{r}$ onto this plane, and $\vec{r}_p + \delta\vec{r}$ is the projection of $\vec{r} + \delta\vec{r}$ onto this plane.

The diagram on the right shows the top view. Vector $\vec{r}_p$ gets rotated to $\vec{r}_p + \delta r$ through an arc, while the vector $\delta\vec{r}$ is a vector that connects the two tips of these vectors.

Let us make two observations from the above figure.

1. Rotation angle $\delta\phi$ can be considered as a vector, $\delta\vec{\phi}$, whose direction is determined by the right handed screw rule. So, the direction of $\delta\vec{\phi}$ is defined as the direction of the rotational axis.
2. For an infinitesimal $\delta\phi$, $\delta\vec{r}$ is perpendicular to both $\delta\vec{\phi}$ and $\vec{r}$, and $|\delta\vec{r}| = |\vec{r}_p|\delta\phi$.

The second point seems intuitive from the figure, but it can also be shown analytically. Noting that the magnitude of a vector is conserved by rotation, we must have $(\vec{r} + \delta\vec{r}) \cdot (\vec{r} + \delta\vec{r}) = \vec{r} \cdot \vec{r}$. Expanding, we get $2\vec{r} \cdot \delta\vec{r} + (\delta r)^2 = 0$, where $\delta r \equiv |\delta\vec{r}|$. Dividing by $2\delta r$, we get $\vec{r} \cdot \vec{e} = -\delta r/2$, where $\vec{e} \equiv \delta\vec{r}/\delta r$ is the unit vector in the direction of $\delta\vec{r}$. So, clearly as $\delta r \to 0$, we get $\vec{r} \cdot \vec{e} = 0$, that is, $\vec{r}$ and $\delta\vec{r}$ are perpendicular to each other.

Since $\delta\vec{r}$ is perpendicular to both $\vec{r}$ and $\vec{\delta}\phi$, one may wonder whether it can be expressed as a vector product. The answer is yes, and most of the proof is already contained in point 2 above, and any necessary filling-in-the-gap is left for your exercise[1].

The change in position vector $\vec{r}$ caused by a rotation by $\delta\phi$ is given by

$$\delta\vec{r} \; = \; \delta\vec{\phi} \times \vec{r}. \tag{10.1}$$

Well, it took some time to derive this simple equation, but we will now see its value.

Suppose that the Lagrangian has a full rotational symmetry. That is, the system has the full isotropy.

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[1] It is useful to resolve the position vector as $\vec{r} = \vec{r}_p + \vec{r}_z$, were $\vec{r}_z$ is on the rotation axis.

Consider the Lagrangian $L(\vec{r}, \dot{\vec{r}}, t)$. Now, consider an infinitesimal rotation by $\delta\vec{\phi}$, an arbitrary but constant vector. From above, we get

$$\delta\dot{\vec{r}} = \delta\vec{\phi} \times \dot{\vec{r}}. \tag{10.2}$$

By infinitesimal rotation, we get

$$\begin{aligned}
\delta L &= L(\vec{r} + \delta r, \dot{\vec{r}} + \delta\dot{r}, t) - L(\vec{r}, \dot{\vec{r}}, t) & (10.3)\\
&= \frac{\partial L}{\partial \vec{r}} \cdot \delta\vec{r} + \frac{\partial L}{\partial \dot{\vec{r}}} \cdot \delta\dot{\vec{r}} & (10.4)\\
&= \frac{\partial L}{\partial \vec{r}} \cdot (\delta\vec{\phi} \times \vec{r}) + \frac{\partial L}{\partial \dot{\vec{r}}} \cdot (\delta\vec{\phi} \times \dot{\vec{r}}) & (10.5)\\
&= \dot{\vec{p}} \cdot (\delta\vec{\phi} \times \vec{r}) + \vec{p} \cdot (\delta\vec{\phi} \times \dot{\vec{r}}) & (10.6)\\
&= \frac{d}{dt}\Big(\vec{p} \cdot (\delta\vec{\phi} \times \vec{r})\Big) & (10.7)\\
&= \delta\vec{\phi} \cdot \frac{d}{dt}\Big(\vec{r} \times \vec{p}\Big). & (10.8)
\end{aligned}$$

For obtaining Eq. 10.6, we used the definition of the linear momentum $\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}}$ (Eq. 9.43), and the Lagrange equation of motion $\frac{\partial L}{\partial \vec{r}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = \dot{\vec{p}}$. For obtaining Eq. 10.8, we used the identity[2] $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$, and took out the time-independent vector $\delta\vec{\phi}$ outside the time derivative.

By rotational invariance, we mean that

$$\delta L = \delta\vec{\phi} \cdot \frac{d}{dt}\Big(\vec{r} \times \vec{p}\Big) = 0 \quad \text{when the system is rotated by } \delta\vec{\phi}. \tag{10.9}$$

If there is a full isotropy, then this must hold, regardless of the direction of $\delta\phi$, while its magnitude is kept a small infinitesimal value. This is possible only if $\frac{d}{dt}\Big(\vec{r} \times \vec{p}\Big) = 0$. So, we get this nice result!

> The full rotational symmetry leads to the conservation of angular momentum vector
>
> $$\vec{L} = \vec{r} \times \vec{p}. \tag{10.10}$$

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[2] Item 7 in page 2 of LN 4.

Tragically, we are using the same symbol L for Lagrangian and angular momentum. Since one is a scalar and the other is a vector, we are good, albeit barely!

At this point, I suggest that a reader pauses and think for a moment. Why is the angular momentum defined as $\vec{r} \times \vec{p}$? Now, we know the answer. The answer is because that is the conserved quantity associated with the rotational invariance[3].

Two important points are to be made.

1. The above arguments are generalized easily[4], if there are many particles involved. In that case, $\delta L = \delta\vec{\phi} \cdot \frac{d}{dt}\Big(\sum_i \vec{r}_i \times \vec{p}_i \Big)$, where $i = 1, ..., N$ is the particle index, assuming that we have N particles. So, we then conclude that **the total angular momentum conservation by the isotropy of space**.

$$\vec{L}_{tot} = \sum_i \vec{r}_i \times \vec{p}_i \quad \text{is conserved by the isotropy of space.} \tag{10.11}$$

2. If the rotational symmetry is not full, but partial, then only the partial component of the total angular momentum is conserved. For example,

$$\left(\vec{L}_{tot}\right)_z = \left(\sum_i \vec{r}_i \times \vec{p}_i\right)_z \quad \text{is conserved by rotational symmetry around the } z \text{ axis.} \tag{10.12}$$

   This is because of the following reason. In the rotational symmetry is restricted to rotation around the z axis, $\delta\vec{\phi}$ cannot be taken to be along any arbitrary direction, but its direction must be fixed along the z direction. So, $\delta L = \delta\phi \frac{d}{dt}\left(\vec{L}_{tot}\right)_z = 0$ means the above.

10.2 Practical summary

In the last section and the last section of the previous lecture note, we have established two important conservation rules. The role of symmetry in these discussions cannot be over-emphasized. Here is a short summary.

1. The conservation of momentum and the conservation of angular momentum are derived from the symmetry principle.

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[3] Similarly, one can ask why the kinetic energy is defined as $\frac{1}{2}mv^2$ in the Newtonian mechanics. The answer is the work-energy theorem.

[4] Left for your exercise.

2. In fact, the symmetry principle says that some quantity is conserved when the space is homogeneous or isotropic. We *define* that conserved quantity as linear momentum or angular momentum, respectively.

In the next section, we shall see that the homogeneity of time (last lecture) leads to a conserved quantity, which we refer to as the **Hamiltonian**. The Hamiltonian is a generalized version of the energy.

Here is a practical summary of what these principles mean. The first two are written in a way that is particularly useful for motions involving a particular linear displacement or a particular angular displacement. In addition to these practical summaries, one must keep in mind that for a closed system the total angular momentum, the total linear momentum, and the total Hamiltonian are always conserved, no matter how many degrees of freedom or how many particles exist in that system.

1. If there is a linear displacement variable, say x, and if the Lagrangian does not depend on it explicitly, i.e., $\frac{\partial L}{\partial x} = 0$, then the linear momentum $p_x = \frac{\partial L}{\partial \dot{x}}$ is conserved.
2. If there is an angular displacement variable, say θ, and if the Lagrangian does not depend on it explicitly, i.e., $\frac{\partial L}{\partial \theta} = 0$, then the angular momentum $p_\theta = \frac{\partial L}{\partial \dot{\theta}}$ is conserved.
3. If the Lagrangian does not depend on t explicitly, i.e., $\frac{\partial L}{\partial t} = 0$, then the Hamiltonian is conserved.

In each of these statements, note that L can depend on other variables still, since we are talking about the vanishing of the partial derivative. For instance, in case 1, the Lagrangian can explicitly depend on $\dot{x}$ and t.

In the discussion of the last section, the angular momentum was defined as $\vec{r} \times \vec{p}$. To show that the z component of this angular momentum is the same as $\frac{\partial L}{\partial \dot{\phi}}$, assuming that ϕ is the azimuthal angle measured in the xy plane, is left for your exercise[5].

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[5] A clean note on this produced by a student will definitely be worth some extra credit points.

10.3 Hamiltonian

The Hamiltonian function is defined as follows.

$$H(p,q,t) \overset{def}{=} p\dot{q} - L(q,\dot{q},t). \qquad p = \frac{\partial L}{\partial \dot{q}} = \text{canonical momentum} \tag{10.13}$$

Or, for a general M degree of freedom problem,

$$H(p_1,...,p_M,q_1,...,q_M,t) \overset{def}{=} \left(\sum_{j=1}^{M} p_j \dot{q}_j\right) - L(q_1,...,q_M,\dot{q}_1,...,\dot{q}_M,t), \tag{10.14}$$

where each p_j is the canonical (conjugate) momentum for q_j, i.e.,

$$p_j \overset{def}{=} \frac{\partial L}{\partial \dot{q}_j}. \tag{10.15}$$

As we have been doing so far, we shall lead most of our discussions assuming the one degree of freedom case, since the generalization to multi-degree-of-freedom case is quite straightforward.

The above definition is what is mathematically referred[6] to as a **Legendre transformation**. What Legendre transformation indicates is a change of independent variables. For instance, in Eq. 10.13, notice that L is written as a function of $q, \dot{q}, t$, while H is written as a function of p, q, t. Why can we say this? The reason can be seen as follows.

$$\begin{aligned}
dH &= d(p\dot{q}) - dL && (10.16)\\
&= \dot{q}dp + pd\dot{q} - dL && (10.17)\\
&= \dot{q}dp + pd\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial t}dt && (10.18)\\
&= \dot{q}dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt && \text{since } p = \frac{\partial L}{\partial \dot{q}} \quad (10.19)\\
&= \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t}dt. && \text{by the Lagrange equation of motion} \quad (10.20)
\end{aligned}$$

The fact that the total differential is expressed as a linear combination of dp, dq, and dt, in this fashion means that p, q, t are the natural choice for independent variables for H.

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[6] Frequently, a Legendre transformation of a function $F(x,y)$ is defined as: $G(x,z) = F(x,y) - zy$ where $z = \frac{\partial F}{\partial y}$. Then, $dG = dF - ydz - zdy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy - ydz - zdy = \frac{\partial F}{\partial x}dx - ydz$. Here, our Legendre transformation is defined with an overall negative sign: $H(x,z) = zy - F(x,y)$, and so $dH = ydz - \frac{\partial F}{\partial x}dx$.

In addition, note that Eq. 10.20 means the following so-called **canonical equations of motion**:

$$\frac{\partial H}{\partial p} = \dot{q}, \tag{10.21}$$

$$\frac{\partial H}{\partial q} = -\dot{p}, \tag{10.22}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{10.23}$$

Of these three equations, the last needs a bit more explanation. That $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ is clear from Eq. 10.20. But, why is it that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$? This is also the property of Eq. 10.20. Divide both sides by dt and notice that the first two terms cancel each other out, leaving exactly $\frac{dH}{dt} = \frac{\partial H}{\partial t}$.

Indeed, Eq. 10.23 is the one that pertains to what is generally referred to as the energy conservation. It has already been advertised in the previous section. Namely, if L does not explicitly depend on t, that is, $\frac{\partial L}{\partial t} = 0$, then what it says is that $\frac{dH}{dt} = 0$: **the Hamiltonian function is conserved**.

The Lagrangian way

Let us summarize the new way of thinking here (as opposed to the old Newtonian way of thinking). First, you must start with the **Lagrangian** L. Without it, you cannot define the action, so it is a no go since the least action principle is THE principle here. For this course, $L = K - U$ is enough. But, perhaps you can keep in the back of your mind that L is not necessarily that in other situations. Often it is the symmetry of the problem (such as the Lorentz invariance) that can guide you to determine what L is. Second, the Lagrangian begets the **canonical conjugate momentum** through the definition $p = \frac{\partial L}{\partial \dot{q}}$. This canonical momentum is conserved if q does not appear in L explicitly, i.e., $\frac{\partial L}{\partial q} = 0$. Third, the **Hamiltonian** is defined as $H = p\dot{q} - L$, and it is conserved if the Lagrangian or the Hamiltonian is not explicitly dependent on time, $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} = 0$. Given L, the Lagrange equation of motion can be used to set up the differential equation to solve. Or, H can be used to set up the canonical equation of motion, in terms of p and q. Either way, note that everything follows from the Lagrangian, L.

Note that, even given that $L = T - U$, the Hamiltonian is generally *not* equal to $T + U$. If the system under consideration is a closed system, then $H = T + U$, but for an open system H does not have to be equal to $T + U$. Even, when H is not equal to $T + U$, it can be conserved. For an open system, it can also happen that $H = T + U$, but it is not conserved. So, in general, you must now consider H according to the definitions, Eq. 10.13 or Eq. 10.14, as the most fundamental. It will make your life easier if you let go of the concept of the energy in place of the Hamiltonian. However, for a closed system, H is always conserved, and that is the ultimate connection of H and the energy. Also, if $L = T - U(q)$, and $T \propto \dot{q}^2$, then $H = T + U$.

Note that the Hamiltonian and the canonical equation of motion provide a natural framework for solving any mechanics problem in the phase space. Indeed, notice that the above canonical equations of motion makes the solution to any mechanical problem a **first order differential equation** in the phase space. That is, given p, q at any point in time (t), we can figure out $p + dp$ and $q + dq$ at the subsequent time $t + dt$ by the canonical equations of motion, since $dq = \frac{\partial H}{\partial p} dt$ and $dp = -\frac{\partial H}{\partial q} dt$. We also know how H itself may change, since $dH = \frac{\partial H}{\partial t} dt$ by Eq. 10.23. So, in this view, solving Newton's equation of motion (LN 2) becomes equivalent to following the trajectory in the phase space (LN 6) according to the canonical equations of motion. This statement is generalized immediately to the case involving any degrees of freedom, M. In the general case, the canonical equations of motion for p and q simply change to

$$\frac{\partial H}{\partial p_j} = \dot{q}_j, \tag{10.24}$$

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j, \qquad j = 1, ..., M. \tag{10.25}$$

while the third equation, Eq. 10.23, remains unchanged.

10.4 Some simple examples

Let us take some simple examples to discuss what the conservation principles mean in practice. Before we go on, let us remember that for a closed system, the total Hamiltonian, the total linear momentum, and the total angular momentum are conserved by the fundamental symmetries of space and time (the translational invariance of time and space, and the rotational invariance of space).

10.4.1 A free particle

Consider a free particle in three dimensions[7].

$$L = \frac{1}{2}m\dot{\vec{x}}\cdot\dot{\vec{x}} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{10.26}$$

Are $H = \dot{\vec{x}}\cdot\vec{p} - L$, $\vec{p} = m\vec{v}$, and $\vec{L} = m\vec{r}\times\vec{v}$, conserved? The answer is yes, yes, and yes.

The answer can be physical and mathematical.

First, the physical answer is that this is a closed system. And so, *assuming* that we defined H, $\vec{p}$, and $\vec{L}$ correctly, they *must* be conserved.

Second, the mathematical answer is as follows. $L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$ is now a function of seven independent variables. Since $\frac{\partial L}{\partial t} = 0$, H is conserved. Since $\frac{\partial L}{\partial x} = 0$, $p_x = mv_x$ is conserved. Similarly, for p_y and p_z. Finally, for $\vec{L}$, one can see that L is invariant under any rotation around the origin, since the kinetic energy is an inner product of two vectors.

10.4.2 A general central force problem

Consider N particles interacting via "central forces." A central force between two particles is a force that can be derived from a potential that depends only on the distance between two particles. For instance, Newton's gravity gives a central force problem, as does Coulomb's law force: $U = A/r$ for two particles at distance r, where A is a constant. Spring force between particles also defines a central force problem: $U = Br^2$ for two particles at distance r, where B is a constant.

A general central force problem can be defined by the Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{x}}_i\cdot\dot{\vec{x}}_i - \sum_{j>i} U_{ij}(|\vec{x}_i - \vec{x}_j|), \tag{10.27}$$

where U_{ij} is the potential energy between particles i and j.

Are $H = \sum_i \dot{\vec{x}}_i\cdot\dot{\vec{p}}_i - L$, $\vec{P}_{tot} = \sum_i m_i\vec{v}_i$, and $\vec{L}_{tot} = \sum_i m_i\vec{r}_i\times\vec{v}_i$, conserved? The answer is, again, yes, yes, and yes.

The answer can be physical and mathematical.

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[7] We consider a non-relativistic particle, as we have been and will be assuming for this course.

First, the physical answer is that this is a closed system. And so, again, they *must* be conserved[8].

Second, the mathematical answer is as follows. $L = L(\{\vec{x}_i\}, \{\dot{\vec{x}}_i\}, t)$ is now a function of $6N + 1$ independent variables, assuming that the spatial dimension is 3. Since $\frac{\partial L}{\partial t} = 0$, H is conserved. For the total linear momentum, $\vec{p}_{tot}$, the easiest way to think about this is to translate *all* vectors $\vec{x}_i$ by an arbitrary but fixed amount. Such transformation leaves L invariant, since (1) it does not affect the derivative, and (2) it does not change the relative position vector $\vec{x}_i - \vec{x}_j$. Lastly, for the total angular momentum, $\vec{L}_{tot}$, all we need to note is that vectors appear only as part of inner products in L: note that $|\vec{x}_i - \vec{x}_j| = \sqrt{(\vec{x}_i - \vec{x}_j) \cdot (\vec{x}_i - \vec{x}_j)} = \sqrt{\vec{x}_i \cdot \vec{x}_i - 2\vec{x}_i \cdot \vec{x}_j + \vec{x}_j \cdot \vec{x}_j}$.

10.4.3 A particle near the surface of the Earth

The Lagrangian is given by

$$L = \frac{1}{2} m \dot{\vec{x}} \cdot \dot{\vec{x}} - mgz = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz, \tag{10.28}$$

where we have taken the vertical direction as the z axis (with up meaning positive).

In this case, H is conserved, as is p_x, p_y, and $L_z = mxv_y - myv_x$. But, p_z, L_x, and L_y are not conserved. The reason is that the system is no longer invariant by translation in the z direction ($z \to z + \Delta$ will change L, where Δ is an arbitrary but fixed number). Also, the system is no longer invariant when rotated around the x axis or the y axis. The kinetic energy part is rotationally invariant, but the potential energy term is not, since the z component of a position vector changes when the vector is rotated around any axis other than the z axis.

This system is not a closed system, since mgz occurs due to the interaction with the Earth, which is left out in the above formulation. Had we included the Earth, then we will find that the total linear momentum and the total angular momentum are all conserved. This fact is in no contradiction to the fact that here we find that $\vec{p}$ and $\vec{L}$ are not completely conserved—they are the quantities for our mass only (not including the Earth).

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[8] It must be emphasized that this is regardless of the form of Lagrangian written. In other words, for a closed system, the Lagrangian would be incorrect, if it did not have this symmetry. Also, note that central forces are not the only possible forces between particles in a general situation.

10.5 Laws and phenomena

Lastly, it must be cautioned that the symmetry of laws and the symmetry of phenomena are quite different things[9].

One can have a rotationally symmetric law, but quite a different phenomena. Think about the shape of the galaxy—it is *not* fully rotationally symmetric. That is, it is not a spherical shape. Think about the universe for that matter. Or, your shape.

Just because there is a translational invariance in time does not mean that things behave the same way at any time[10]. Clearly things *happen* every moment. As another example, that time dilates in a moving frame is perfectly consistent with time-invariant laws of physics.

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[9] At the quantum level, however, one finds that the symmetry of laws must be reflected in the symmetry of phenomena. The case when this is not the case is referred to as the "broken symmetry," a very important concept that became recognized in the latter half of the last century.

[10] You might be amused, however, to note that there are some cosmologists, who think that the concept of time can disappear in the ultimate quantum model of the Universe. Some of them might be saying that really nothing is happening in this Universe. The fact is that physicists do not yet understand time well at all. Read about the issue of time in, for example, Lee Smolin, Physics Today, 67, 38 (2014).