Notes for Lecture 9

Lagrangian, Symmetry, Conservation

Here, the Lagrangian mechanics is derived from the principle of least action. We will also discuss the important subject of symmetry and conservation in the context of the Lagrangian mechanics.

9.1 Lagrangian mechanics

Let us recall, from the last lecture, the principle of least action. For a generalized coordinate $q(t)$, and the Lagrangian $L \equiv K - U$, we consider the initial points of time and coordinate, t_1 and $q(1) \equiv q(t_1)$, and the final points of time and coordinate, t_2 and $q(2) \equiv q(t_2)$. Out of all mathematical paths going from $q(1)$ to $q(2)$, the physical path[1] consistent with the given L, Lagrangian, is determined by

$$\delta S = \delta \int_1^2 L(q, \dot{q}, t) dt = 0 \tag{9.1}$$

to first order in variation $\delta q(t)$ of $q(t)$.

It is important to note that all mathematical paths that we consider start and end at the same points in time and space, and therefore $\delta q(1) = \delta q(2) = 0$ at the two end points.

How does L change when $q(t) \to q(t) + \delta q(t)$? For simplicity, from now on, we will

[1] Or, physical paths, since, in general, multiple extrema of S can occur.

write q for $q(t)$ and δq for $\delta q(t)$. From multi-variable calculus

$$\delta L = \frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\delta \dot{q}. \tag{9.2}$$

Also, note that $\delta \dot{q} = \frac{d\delta q}{dt}$. So, we get

$$\delta S = \int_1^2 dt\left(\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\frac{d}{dt}\delta q\right) \tag{9.3}$$

$$= \int_1^2 dt\left(\frac{\partial L}{\partial q}\,\delta q - \delta q\,\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_1^2 \qquad \text{integration by parts} \tag{9.4}$$

$$= \int_1^2 dt\left\{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right\}\delta q(t). \qquad \text{since } \delta q(1) = \delta q(2) = 0 \tag{9.5}$$

$$\tag{9.6}$$

Clearly, if we impose $\delta S = 0$, for any $\delta q(t)$ that is small but of arbitrary shape, then the expression inside the curly brackets *must vanish uniformly at any point in the considered time interval*. An easy way to see this is to consider $\delta q(t)$ a tiny "blip function" that is zero at all times except in a very small vicinity of t_m. We can make the time duration of that blip much smaller than any physical time resolution, while keeping the blip mathematically smooth. For such a blip function $\delta q(t)$, the above integral is given by $\approx \{...\}_{t=t_m}\,\delta q(t_m)\varepsilon_t$, where ε_t is the time duration of that blip. So, we get $\{...\}_{t=t_m} = 0$. Since t_m can be an arbitrary value $t_1 \leq t_m \leq t_2$, we have proved that (after multiplying the expression in the curly brackets by -1)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \qquad \textbf{Lagrange equation of motion}. \tag{9.7}$$

This equation of motion is also referred to as the Euler-Lagrange equation.

So far, we have been considering only one generalized coordinate $q(t)$. However, for a general case where there are M degrees of freedom, we must recognize that

$$L = L(q_1, ..., q_M, \dot{q}_1, ..., \dot{q}_M, t). \tag{9.8}$$

The above Lagrange equation of motion can be generalized readily to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad \textbf{Lagrange equation of motion},\ i = 1, ..., M. \tag{9.9}$$

The proof for this generalization is left for your exercise.

Before we go further, we note that the principle of least action does not assume any particular form of K and U, in $L = K - U$. Of course, the familiar forms of $K = \frac{1}{2}mv^2$ and $U = U(\vec{r})$ are often valid in the case of single particle dynamics, but this is *not always the case*. For instance, if a charged particle is moving in a magnetic field, then the potential energy depends on $\vec{v}$.

9.2 Newton's laws from the Lagrangian point of view

Let us consider a one dimensional particle problem, where a particle of mass m is moving in linear space x ($q = x$) with a potential energy $U(x)$.

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - U(x). \tag{9.10}$$

Let us see what the Lagrangian equation of motion looks like.

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}} &= \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \\ \frac{\partial L}{\partial q} &= \frac{\partial L}{\partial x} = -\frac{dU}{dx}, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} &= m\ddot{x} + \frac{dU}{dx} = 0. \end{aligned}$$

Clearly, this is just the familiar Newton's laws: $m\ddot{x} = F(x)$, since $F(x) = -\frac{dU}{dx}$.

9.3 Generalized force

By design, the Lagrangian mechanics is "clean," in the sense that no non-mechanical elements are introduced. For instance, resistive forces such as air resistance and friction are not considered, as explained in the last lecture. The reason for this is that these forces are not fundamental forces but they are the consequences of many body dynamics. These resistive forces are *generated* from the "clean" Lagrangian with many degrees of freedom due to many particles involved, through the consideration of the entropy[2].

So, in a way, it is nice not to include those resistive forces. On the other hand, it could be regarded as a practical limitation of the Lagrangian formalism not to include those forces that do exist in the real world. No worries. For those forces that are of "empirical resistive kind," we can put them in by hand as follows

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i, \qquad Q_i\text{: \textbf{generalized force}}. \tag{9.11}$$

[2]This description does *not* mean that any physicist has actually proven *how* this emergence process occurs!

The reason that Q_i is called the generalized force is this. If q_i represents a linear displacement, then Q_i will be a force. If q_i represents an angular displacement, then Q_i will be a torque due to resistive force. So, it is not always a force.

In this course, we will stick with the clean version of the Lagrangian mechanics, where we do not put any empirical generalized force in by hand. However, we shall see that the generalized force concept is still useful to consider, when we consider a Lagrange problem with constraints.

9.4 The Lagrangian way

From the Lagrangian mechanics point of view, we no longer ask "what is the force acting on the particle?". Rather we ask the following questions.

1. What are the generalized coordinates?
2. What is the Lagrangian, L? Make sure to remember that t, q, and $\dot{q}$ are to be treated as independent variables of $L(q, \dot{q}, t)$.

Once we get answers to these[3], the rest is "just math."

In setting up the Lagrangian and solving the problem, the following point becomes handy quite often. If you set up a Lagrangian and then add any differentiable function of t, say, $f(t)$, to it, the same Lagrangian equation of motion will be obtained, and there is no dynamical consequence. Clearly, this is the case since the Lagrangian equation will completely ignore such an addition of $f(t)$: $L(q, \dot{q}, t) \to L(q, \dot{q}, t) + f(t)$ will not change the equation of motion, since $\frac{\partial f}{\partial q} = 0$ and $\frac{\partial f}{\partial \dot{q}} = 0$. This observation can be generalized as the first point below.

1. Adding a total time derivative, $\frac{dF(q,t)}{dt}$, to the Lagrangian results in no change in physics:

$$L(q, \dot{q}, t) \;\to\; L(q, \dot{q}, t) + \frac{dF(q,t)}{dt}. \tag{9.12}$$

2. Scaling the Lagrangian by a constant ($L \to CL$, where non-zero constant C does not depend on any of $q, \dot{q}$, and t) does not change any physics.

[3]We might also want to include generalized forces, after setting up the Lagrangian equation of motion, as explained in the previous section.

The first point[4] is a direct consequence of the principle of least action. On adding the function $dF(q,t)/dt$, we see that $\delta S \to \delta S + \delta \int_1^2 dF(q,t) = \delta S + \delta F(2) - \delta F(1)$, where $\delta F(1) \equiv \delta F(q(1), t_1)$ and $\delta F(2) \equiv \delta F(q(2), t_2)$. But, $F(q,t)$ cannot change at end points, since by definition we have all of t_1, t_2, $q(1)$, and $q(2)$ fixed, for any variation $\delta q(t)$. As F is a function of q, t only by assumption, its variation at end points must be zero, then: $\delta F(2) = \delta F(1) = 0$. So, δS does not change at all as we add $dF(q,t)/dt$, and neither does the physical path that minimizes the Lagrangian. We get the same equation of motion.

The first point explains the well-known fact that adding a constant to the potential energy does not change any physics, but it goes well beyond such simple piece of physics.

The second point[5] can be understood from the same variational principle, or simply from the Lagrangian equation of motion itself, which clearly remains unchanged if the whole equation is multiplied by a non-zero constant C.

Both these facts can be used frequently in the actual problem solving.

9.5 Simple pendulum examples

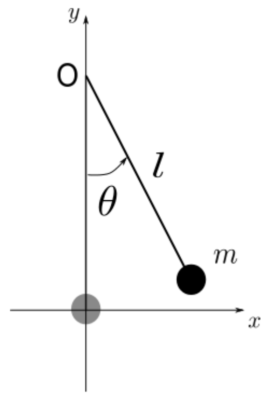
Let us consider couple of simple examples involving a simple pendulum.

9.5.1 Simple pendulum

For a simple pendulum with mass m and length l, how would we approach the problem from the Lagrangian mechanics point of view?

[4]The term $dF(q,t)/dt$ can be called a "gauge term" since it is through this term that the gauge invariance gets incorporated.

[5]Here, one might question, if L is given, *why* would we want to scale it by a constant? An answer is that we would do so if we change the unit of the energy. Or, we could be changing the units of time and space in such a way that the Lagrangian is simple scaled by a constant.



First, the generalized coordinate may be taken as the angle θ (see the diagram; the pivot point is O). Second, what is the Lagrangian? For the kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$. For the potential energy, $U = -mgl\cos\theta$ up to a constant offset.

So,

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta. \tag{9.13}$$

The rest is just math! Using Eq. 9.7, with $q = \theta$, we get

$$ml^2\ddot{\theta} = -mgl\sin\theta. \tag{9.14}$$

Note that we did not have to think about any vector product and the torque (and its sign), and we just got the correct torque on the right side of this equation.

9.5.2 Simple pendulum in an accelerating car

Take the same simple pendulum and put it in a car, which we assume is accelerating with a constant acceleration vector $\vec{a}$ along the horizontal direction.

First, we would still use only θ for our generalized coordinate. Second, what is the Lagrangian? For this, we just have to make some simple coordinate transformations, and we will be able to write down what L is.

But first of all, we better remember that we should measure everything from an inertial frame, and so we must consider the "lab frame," not the reference frame inside the car, as our reference frame. Here, the "lab frame" is the street reference frame, which we assume is inertial. When observed from the lab frame, the car is going at velocity $v = v_0 + at$ along the x direction, which we take to be along the horizontal direction. Thus, we get the following coordinate transformation

$$X = x + v_0 t + \frac{1}{2}at^2, \tag{9.15}$$

$$Y = y, \tag{9.16}$$

where the small case coordinate letters (x, y) are for the car frame, and the large case coordinate letters (X, Y) are for the lab frame. When observed in the car, the position of the pendulum is given by (see the diagram above for the previous example)

$$x = l\sin\theta, \tag{9.17}$$

$$y = l(1 - \cos\theta). \tag{9.18}$$

The Lagrangian is readily given by

$$L = K - U = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) - mgY. \tag{9.19}$$

Note that

$$\dot{X} = \dot{x} + v_0 + at = l\dot{\theta}\cos\theta + at + v_0, \tag{9.20}$$

$$\dot{Y} = \dot{y} = l\dot{\theta}\sin\theta. \tag{9.21}$$

Plugging these into the expression for L, and after some algebra, we get

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + ml(v_0 + at)\dot{\theta}\cos\theta + \frac{1}{2}m(v_0 + at)^2 + mgl\cos\theta,$$

where we ignored an additive constant $-mgl$. Also, note that the third term is a function of time only, and so we can completely ignore it. So, we get

$$L = ml\left(\frac{1}{2}l\dot{\theta}^2 + (v_0 + at)\dot{\theta}\cos\theta + g\cos\theta\right). \tag{9.22}$$

Now, we can proceed to obtain the equation of motion. For doing so, we shall use the scaled Lagrangian $L' = L/(ml)$.

$$\frac{\partial L'}{\partial\dot{\theta}} = l\dot{\theta} + (v_0 + at)\cos\theta,$$

$$\frac{d}{dt}\frac{\partial L'}{\partial\dot{\theta}} = l\ddot{\theta} + a\cos\theta - (v_0 + at)\dot{\theta}\sin\theta,$$

$$\frac{\partial L'}{\partial\theta} = -(v_0 + at)\dot{\theta}\sin\theta - g\sin\theta.$$

So, the equation of motion is

$$\ddot{\theta} = -\frac{1}{l}\Big(g\sin\theta + a\cos\theta\Big). \tag{9.23}$$

Note that while v_0 was kept along in our calculation, it drops out of the equation of motion in the end. This is due to the Galilean invariance. Anticipating, we could have set $v_0 = 0$ from the beginning. In any case, we get two torques, one from the gravitational force, and the other from the acceleration a (due to a fictitious "inertial force" $-ma$).

Physics in a non-inertial reference frame

If our world was the car in the above problem, and if we assume that we never get out of the car, then we would feel that the inertial force $-ma$ is always present, and we would not be able to distinguish it from a real gravitational force. So, the total gravitational force that we feel will be due to two gravitational potential energies, mgy and max. The Lagrangian that we will set up is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - m(ax+gy) = \frac{1}{2}ml^2\dot{\theta}^2 - ml(a\sin\theta - g\cos\theta). \tag{9.24}$$

It can be shown that this Lagrangian (1) gives the same equation of motion as Eq. 9.23 and (2) differs from Eq. 9.22 by a total time derivative, $dF(\theta,t)/dt$. It is left for your exercise to find that $F(\theta,t)$!

Let us think about what Eq. 9.23 means. First, in equilibrium, we must have $\ddot{\theta} = 0$. Thus, the equilibrium angle is given by

$$g\sin\theta_{eq} + a\cos\theta_{eq} = 0, \tag{9.25}$$

$$\tan\theta_{eq} = -\frac{a}{g}, \tag{9.26}$$

$$\sin\theta_{eq} = -\frac{a}{\sqrt{a^2+g^2}}, \tag{9.27}$$

$$\cos\theta_{eq} = \frac{g}{\sqrt{a^2+g^2}}. \tag{9.28}$$

So, at equilibrium, the pendulum is not vertical, but, at a negative angle, assuming a is positive. In any case, that θ_{eq} has the opposite sign to a is a well understood fact, if you remember that your body feels a push to the back when you are accelerating your car forward. This is of course the so-called inertial force that is felt in an accelerating frame.

Next, we can ask, what is the nature of the motion around θ_{eq}? To figure it out, we can expand $g\sin\theta + a\cos\theta$ around θ_{eq}.

$$\begin{aligned}
\sin\theta &= \sin(\theta_{eq} + \xi) && \xi \equiv \theta - \theta_{eq} \\
&= \sin\theta_{eq}\cos\xi + \cos\theta_{eq}\sin\xi \\
&\approx \sin\theta_{eq} + \cos\theta_{eq}\,\xi, && \text{for small oscillation } |\xi| \ll 1 \\
\cos\theta &= \cos(\theta_{eq} + \xi) \\
&= \cos\theta_{eq}\cos\xi - \sin\theta_{eq}\sin\xi \\
&\approx \cos\theta_{eq} - \sin\theta_{eq}\,\xi. && \text{for small oscillation } |\xi| \ll 1
\end{aligned}$$

Therefore, Eq. 9.23 is approximated, for small oscillation, as

$$\ddot{\xi} = -\frac{1}{l}\left(g\sin\theta_{eq} + a\cos\theta_{eq} + g\xi\cos\theta_{eq} - a\xi\sin\theta_{eq}\right) \tag{9.29}$$

$$= -\frac{1}{l}\left(g\cos\theta_{eq} - a\sin\theta_{eq}\right)\xi \qquad \text{using Eq. 9.25} \tag{9.30}$$

$$= -\frac{\sqrt{g^2 + a^2}}{l}\,\xi. \qquad \text{using Eqs. 9.27, 9.28} \tag{9.31}$$

So, we get a simple harmonic oscillation, which is just what we have anticipated[6], and the natural frequency is given by

$$\omega_0 = \frac{\sqrt[4]{g^2 + a^2}}{\sqrt{l}}. \qquad \omega_0^2 = \frac{\sqrt{g^2 + a^2}}{l} \tag{9.32}$$

9.6 Symmetry principles

Symmetry is important. It would appear that its importance cannot be overemphasized. A well-known Nobel laureate theorist once said that about three quarters of all physics is about symmetry[7].

The symmetry and conservation principle may sound difficult. It really is not.

The reason why it sounds difficult may be a "language issue." Symmetry is usually discussed in such grand words as "the homogeneity of time and space" and "the isotropy of space," and more. And, the symmetry can be discussed and analyzed at length by the wonderful mathematics that is the group theory. If you have not gotten

[6] The motion around any "usual" stable equilibrium point is a simple harmonic motion. Here, "usual" means that the second derivative of the potential energy is positive, not zero.

[7] To be more precise, what is important is how symmetry breaks (spontaneous symmetry breaking) as much as how symmetry leads to conserved quantities.

on top of the group theory, then you may feel a bit weak about all these talks about symmetry and related issues.

But let us ask what it is really ... apart from all those grand words and all those mathematical theories (which by the way is really useful and which I hope you will learn someday).

Despite the facade built by grand words of physics and mathematics, the essential physics about symmetry can be viewed as really simple.

Say, you do an experiment of some kind. On a table top. Perhaps you are colliding two balls. Or, you are investigating some chemical reaction (which, by the way, is really outside the realm of classical mechanics, but that is alright—symmetry is applicable in all physics). Let us assume that your experiment does not depend on any other thing than what is on the table. It is not affected by the Earth's spinning, the Earth magnetic field, etc. Also, it does not depend on whether you turn on the room light or not. In other words, we can say your experiment on that table forms a "closed system." This is our assumption, which would be satisfied reasonably well by many kinds of experiments.

Now, suppose you do the experiment today, and you do it again tomorrow under identical conditions. You would expect that the experiment will give exactly the same result. Well, that may be slightly misleading. This is what I mean. Any experiment will be affected by statistical fluctuations and you can never obtain exactly the same result in the literal sense. However, when those fluctuations are averaged out, your result of today should agree with your result of tomorrow. Let us say that that is what we actually mean by "getting the same result." This is the symmetry that we call **"the homogeneity of time"**.

It means that physical laws should not change over time. This is true as far as we know.

Now, you imagine doing the same experiment, but do it in one corner of the room and then do it in another corner of the room, after simply translating the table, that is after simply rolling the table with all stuff on it intact. Again, you would expect to get the same result in the new corner, as you did in the first corner. You would expect, in fact, that the experiment will work no matter where you put the table. This is the symmetry that we call **"the homogeneity of space"**.

It means that physical laws should not change just because we simply moved from point 1 to point 2 in the Universe. This is true as far as we know.

Physicists attribute this particular symmetry to Newton, who had this legendary moment of epiphany when he realized that maybe the apple falling from a tree works the same way as the Moon going round the Earth. What a breathless moment he must have had on such a moment of insight!

Finally, let us imagine that you do an experiment. And then you rotate the whole set up by some angle and then do the experiment again. This time too you would expect that you get the same result. This is the symmetry that we call **"the isotropy of space"**.

These are some of the basic symmetries that we physicists have learned as fundamental principles of Nature.

There are other symmetries, of course. Many more, in fact. But we won't need them here in this course. Just one more example. The theory of relativity is another famous theory about symmetry—symmetry of moving reference frames. Indeed, Einstein is the one who gets credit for putting the symmetry principle at the forefront of physics, through his very successful theories of relativity.

So, the concept of symmetry is quite plain and simple. But let us mention one more thing. A very important thing. You probably know that science works only so far as experiments are reproducible. If the results of an experiment on a closed system depend on time and space, then it means that group A and group B who perform the same experiment can never compare their notes. [In such an experiment, one might suspect that the assumption of a "closed system" is incorrect.] Such a situation forbids scientific progress of any kind. The point here is this. The concept of symmetry is not an idle, abstract, or vacant, one at all. *If these symmetries that we discussed above did not exist, then science as we know it would not exist.*

9.7 Conservation principles

Another reason that symmetry is important is because it leads to conservation principles. **This is true in all physics, not just in classical mechanics.** Here, we consider the consequences of the three kinds of symmetry mentioned above, within the realm of classical mechanics.

Let us pause here one minute, though. Let us think—what does it mean when we say that "we should expect the same experimental result" when we do the experiment at a different time, at a different position, or at a different orientation in space? It means that, despite the difference in time, or the different in position, or the difference in orientation, if we prepare the experiment the same way, i.e. if we give the same initial conditions, then the final outcome must be the same. This can only be possible if the mechanical law is invariant under the corresponding "symmetry operations," translation in time or space or rotation in space. In other words, for a closed system, we require that Newton's laws are invariant on translation in time or space or rotation in space. But, as we all know, the Lagrangian is a much nicer scalar quantity from which Newton's law can be derives, and so we might as well say the following.

For a closed mechanical system, the Lagrangian is invariant on translation in time, on translation in space, or on rotation in space. A closed system means a system of particles that interact only within themselves. We attribute this to the homogeneity of time, the homogeneity of space, and the isotropy of space, respectively.

One should note if the Lagrangian is invariant then the equation of motion is invariant too, since the Lagrangian equation of motion is derived directly from the Lagrangian.

When the Lagrangian is invariant under a "symmetry operation" (such as translation in time or space, or rotation in space), a conserved quantity is guaranteed (energy, momentum, or angular momentum, respectively for these examples). Such conserved quantities should exist, then, trivially for closed systems. But this is not all. **Even for an open system**, a certain symmetry can exist in full or in part. Then, the corresponding conserved quantity follow. This property generally goes by the name of "Noether theorem."

9.8 Momentum conservation

Let us consider the **"translational symmetry in space"** first. Or, the symmetry that we called **the homogeneity of space**.

Consider transforming the coordinate system such that each position vector is transformed as

$$\vec{r} \;\to\; \vec{r} + \delta\vec{r} \tag{9.33}$$

where $\delta\vec{r}$ is a constant, but arbitrary, vector. This means $\dot{\vec{r}}$ does not change. The Lagrangian $L = L(\vec{r}, \dot{\vec{r}}, t)$ changes as follows

$$L(\vec{r}, \dot{\vec{r}}, t) \;\to\; L(\vec{r} + \delta\vec{r}, \dot{\vec{r}}, t) = L(\vec{r}, \dot{\vec{r}}, t) + \frac{\partial L}{\partial \vec{r}} \cdot \delta\vec{r} \tag{9.34}$$

where

$$\frac{\partial}{\partial \vec{r}} \stackrel{def}{=} \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}. \tag{9.35}$$

Now, we can apply the Lagrange equation

$$\frac{\partial L}{\partial \vec{r}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} \tag{9.36}$$

where

$$\frac{\partial}{\partial \dot{\vec{r}}} \stackrel{def}{=} \hat{x}\frac{\partial}{\partial \dot{x}} + \hat{y}\frac{\partial}{\partial \dot{y}} + \hat{z}\frac{\partial}{\partial \dot{z}} \tag{9.37}$$

to conclude that

$$L(\vec{r}, \dot{\vec{r}}, t) \rightarrow L(\vec{r} + \delta\vec{r}, \dot{\vec{r}}, t) = L(\vec{r}, \dot{\vec{r}}, t) + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}}\right) \cdot \delta\vec{r}. \tag{9.38}$$

Canonical momentum: for a given generalized coordinate q, the quantity

$$p_q \stackrel{def}{=} \frac{\partial L}{\partial \dot{q}} \tag{9.39}$$

defines the **canonical momentum** for q.

Now, from the homogeneity of space, we must require that L is invariant under translation in space by $\delta\vec{r}$. Namely, in Eq. 9.38, we require that

$$L(\vec{r} + \delta\vec{r}, \dot{\vec{r}}, t) = L(\vec{r}, \dot{\vec{r}}, t), \qquad \text{for } \textit{any } \delta\vec{r}. \tag{9.40}$$

This is possible if and only if

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = 0. \tag{9.41}$$

In other words,

$$\frac{d\vec{p}}{dt} = 0, \tag{9.42}$$

where

$$\vec{p} \equiv \frac{\partial L}{\partial \dot{\vec{r}}} \quad \text{is the } \textbf{linear momentum}. \tag{9.43}$$

It is left for your exercise to show that $\vec{p}$ defined this way is indeed $m\vec{v}$ if the kinetic energy is given by $\frac{1}{2}mv^2$ and the potential energy is independent of $\vec{v}$.

Now, in the above, we considered only one particle. What if we consider N particles? In that case, the same argument as above can be used. It is left for your exercise to show that, in the general case, it is the **total linear momentum** $\vec{P}_{tot} = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{\vec{r}}_i}$ that is conserved by the translational invariance. Of course, the total momentum is given by $\vec{P}_{tot} = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i$ if the kinetic energy is given by $\frac{1}{2}mv^2$ for each particle.

Take a deep breath and take in what we discovered here! The reason why the momentum is conserved is because the space is homogeneous, namely experiment should be reproducible, whether it is performed here, on the east coast, or on Mars, assuming that we prepare the environment of the experiment exactly the same way.

What is more is that the conservation law applies, even if the translational invariance is partially true. For instance, let us consider the Lagrangian for a particle in the constant gravitational field near the surface of the Earth. The potential energy is mgy, and the Lagrangian is given by $L(\vec{r}, \vec{v}, t) = \frac{1}{2}mv^2 - mgy$. Notice that, if the space is translated in the x direction ($x \to x + \delta x$, where δx is a constant) or the z direction ($z \to z + \delta z$, where δz is a constant), the Lagrangian does not change, since L does not depend on x or z and $v^2 = v_x^2 + v_y^2 + v_z^2$ does not change if $x \to x + \delta x$ where δx is a constant, or $z \to z + \delta z$ where δz is a constant. However, clearly, L is not invariant if $y \to y + \delta y$, since L does depend on y. The result? p_x and p_z are conserved, but p_y is not.