Notes for Lecture 8

The principle of least action

Here, we re-formulate Newtonian mechanics from a different point of view. This point of view is better in many ways.

8.1 The principle

Maybe a catchy name: the **principle of least action (PoLA)**. It is also called **Hamilton's principle**.

In this course, this principle is completely equivalent to Newton's laws. From a more general perspective, it is easily argued that this principle seems "deeper" or "more elegant" than Newton's law, because it can be applied to other branches of physics as well, like optics, electro-magnetics, and the relativity, when Newton's laws as we learned them becomes impossible, or cumbersome according to our current knowledge, to generalize. For instance, the well-known **Fermat's principle**, that the light travels in a path that minimizes the time, can be derived from the principle of least action in optics [1]. Also, it is much clearer to see "how quantum mechanics arises" in this view (especially in Feynman's view of quantum mechanics), while, in general, the same task is viewed as impossible if one starts from Newton's laws point of view.

The PoLA means that the following integral, the so-called **action**,

$$S[q(t)] = \int_1^2 dt\, L(q(t), \dot{q}(t), t) \tag{8.1}$$

[1] This may be treated in a homework problem.

is **stationary** when $q(t)$ is the actual motion. Let me analyze this sentence one by one. But, first of all, note that the integration range is written as $\int_1^2$, to mean going from t_1 to t_2, not from value 1 to value 2.

1. $S[q(t)]$ defines a **functional**. The square bracket [ ] is used to emphasize the functional nature of S. It means that S takes a function $q(t)$ as input and returns a number as output. In contrast, a function takes a number and returns a number. **For Hamilton's principle, only those $q(t)$'s for which $q(t_1) \equiv q(1)$ and $q(t_2) \equiv q(2)$ are fixed are considered[2].** $q(1)$ and $q(2)$ are initial and final positions, in terms of the "generalized coordinate" q. In the integral, $q(t)$ can be any trajectory, physical or hypothetical. Having clearly understood that S is a functional, please do not be surprised, should you see $S(q)$ instead of $S[q]$, when the context is clear that S is a functional.

2. $q(t)$ is a **generalized coordinate**. Here, we only consider one degree of freedom, and so there is only one q. Soon, we will consider many degrees of freedom, in which case we will consider q_i's. The most common example of q is x (or y or z). Or, the angle θ, as in the simple pendulum problem. In general, q can be any function of the linear or the angular coordinates, time and any other parameters of the problem, as long as $L = L(q, \dot{q}, t)$, i.e. the system is fully specified by the generalized coordinate, its time derivative and the time. **So, there will be as many generalized coordinates as the degrees of freedom.** As such, q does not need to have the dimension of length. It can be dimensionless (e.g. angle), or it may have other dimensions (the dimension of the momentum, the angular momentum, the energy, etc). A clever choice of q can make a certain property of the problem obvious[3].

3. L is the so-called **Lagrangian**. It is defined as

$$L \stackrel{def}{=} K - U, \tag{8.2}$$

   where K is the kinetic energy and U is the potential energy.

4. That $S[q(t)]$ is stationary means the following. Suppose that the *true* motion occurs as $q_T(t)$. Now, imagine adding a small **variation** $\delta q(t)$ to $q_T(t)$. Consider the subsequent change of S, $\delta S[q_T(t)] \stackrel{def}{=} S[q_T(t)+\delta q(t)] - S[q_T(t)]$. That $S[q(t)]$ is stationary at $q_T(t)$ means that

$$\delta S[q_T(t)] = 0, \qquad \text{for any } \delta q(t). \tag{8.3}$$

[2] Perhaps some justification must be made for notations $q(1)$ and $q(2)$, which are, perhaps, not as elegant as q_1 and q_2. The reason why we do not use the latter is because they are reserved for describing different degrees of freedom.

[3] Look for an example in homework.

As we are just beginning this topic, here we distinguished between the general $q(t)$ and the true motion $q_T(t)$. From now on, however, we will not be making such distinctions, assuming that the context makes it clear when $q(t)$ is an arbitrary[4] one, and when $q(t)$ is the true motion.

8.1.1 Stationary?!

In Eq. 8.3, we wrote $\delta S = 0$, as though it should be obvious to any one what it means, but it is clearly a strange expression. What it actually means is that

$$\delta S = O(\delta q^2) \qquad \text{for any small } \delta q. \tag{8.4}$$

Namely, what it really means is that δS vanishes in the *first order* of the perturbation δq.

Often this is rewritten as

$$\frac{\delta S}{\delta q} = 0, \tag{8.5}$$

where the left hand side is a functional derivative (see Section 8.3 in case you immediately begin to wonder what in the world a functional derivative may mean).

Having explained what $\delta S = 0$ means and knowing that you are familiar with multi-variable calculus (see Section 8.3) and perturbation, I hope that you will agree that it is at least not too crazy to say $\delta S = 0$, as physicists and engineers routinely do, when what we really mean is that the first order response of δS to δq is zero.

8.1.2 Example—free particle

Consider the motion of the free particle, $x = x_0 + v_0 t$. Consider $q(1) = x(1) = 0$ and $q(2) = x(2) = v_0$ (and so we have chosen $t_1 = 0$ and $t_2 = 1$, in some unspecified unit). Show that $\int_1^2 L dt$ is indeed minimum for the true path, by examining the integral for $x = x_0 + v_0 t + f(t)$, where $f(t)$ is any function that satisfies $f(1) = f(2) = 0$. Also, show that S is stationary at the physical motion: $\delta S = 0$ for any small variation $f(t)$.

<u>Solution</u> ✎ `The action integral` $S = \int K dt$`, since there is no potential.`

[4] Arbitrary, as long as it is reasonably "nice." It must have a continuous first derivative

For the given x, we get $v = v_0 + \dot{f}(t)$. And so,

$$\begin{aligned} S &= \frac{m}{2}\int_1^2 v^2 dt \\ &= \frac{m}{2}\int_1^2 \left(v_0 + \dot{f}(t)\right)^2 dt \\ &= S_0 + m v_0 \int_1^2 \dot{f}(t) dt + \frac{m}{2}\int_1^2 \left(\dot{f}(t)\right)^2 dt \end{aligned}$$

where $S_0 \equiv \frac{m}{2}\int_1^2 v_0^2 dt = \frac{m v_0^2}{2}(t_2 - t_1)$. Notice that the second term is zero since the integral for that term is $\int_1^2 \dot{f} dt = \int_1^2 df = f(2) - f(1)$, and since $f(1) = f(2) = 0$ by assumption. So,

$$S = S_0 + \frac{m}{2}\int_1^2 \left(\dot{f}(t)\right)^2 dt$$

Now, the second term is clearly positive for any non-constant function $f(t)$. And note that due to the boundary condition $f(1) = f(2) = 0$, any constant function $f(t)$ is identically zero. Therefore, we see that for any non-zero $f(t)$, we get

$$S > S_0.$$

We have just proved explicitly that the least action principle works for the free particle case!

Now, let us prove that $\delta S = 0$ for any small variation $f(t)$. For any given function $f(t)$ of any (differentiable) shape, we can introduce $g(t) = Af(t)$ (A is a real number) and use this as a variation. We shall assume that $f(t)$ is independent of A, which simply acts as a scaling factor. Now, replacing $f(t)$ in the above result for S with $g = Af$, we get

$$S = S_0 + A^2 \frac{m}{2}\int_1^2 dt (\dot{f}(t))^2.$$

Now, consider turning on A from 0 to a very small amount, δA. The variation of S due to δA is given by

$$\delta S = \delta A^2 \frac{m}{2}\int_1^2 (\dot{f}(t))^2.$$

This is second order in δA, which proves that S is stationary: $\delta S = 0$ in the meaning discussed in Section 8.1.1.

8.1.3 Example—SHO

Consider a simple harmonic oscillator with $\omega_0 = 1$. Choose $t_1 = 0$, $t_2 = 2\pi$, $q(1) = x(1) = 0$, $q(2) = x(2) = 0$. In this case, the physical solution can be taken as $\sin t$.

Choose *one* differentiable function $f(t)$ (which is not proportional to $\sin t$) that satisfies $f(1) = f(2) = 0$. Show that $S[\sin t + Af(t)]$ is stationary with respect to A: i.e., $\frac{dS}{dA} = 0$.

<u>SOLUTION</u> ✎ Let us consider the following variation $Af(t) = At(t-2\pi)$, where A is any non-zero real number. Note that in this case we have the potential energy as well as the kinetic energy.

$$\begin{aligned} S[x(t)] &= \int_1^2 \left(\frac{1}{2}mv^2 - \frac{1}{2}kx^2\right) dt \\ &= \frac{m}{2}\int_1^2 \left(v^2 - x^2\right) dt \qquad \because \omega_0 = 1 \\ &\equiv \frac{m}{2} I. \qquad I \equiv \int_1^2 (v^2 - x^2) \end{aligned}$$

We shall examine $I \equiv \frac{2S}{m}$ from now on. Using $x = \sin t + Af(t) = \sin t + At(t - 2\pi)$, we get $v = \cos t + A\dot{f} = \cos t + 2A(t - \pi)$.

$$\begin{aligned} I &= \int_0^{2\pi} dt \left[\cos^2 t + 4A(t-\pi)\cos t + 4A^2(t-\pi)^2 - \sin^2 t - 2At(t-2\pi)\sin t - A^2t^2(t-2\pi)^2\right] \\ &= I_0 + \int_0^{2\pi} dt \left[4A(t-\pi)\cos t - 2At(t-2\pi)\sin t + 4A^2(t-\pi)^2 - A^2t^2(t-2\pi)^2\right] \end{aligned}$$

where $I_0 = \int_0^{2\pi} dt(\cos^2 t - \sin^2 t) = 0$ is the value for the SHO motion. In the remaining integral, both the first term and the second term in the angular bracket integrate to zero, since each of them is an odd function of $\xi \equiv t - \pi$.

$$\begin{aligned} I &= A^2 \int_0^{2\pi} dt \left[4(t-\pi)^2 - t^2(t-2\pi)^2\right] \\ &= A^2 \frac{8}{15}\pi^3(5 - 2\pi^2) \approx -244\, A^2 < 0. \end{aligned}$$

Clearly, $\frac{dI}{dA} = 0$, since $I \propto A^2$. For the quadratic variational function that we have chosen, we see that the action is the maximum for the actual motion. Let us consider another example. $Af(t) = A\sin(2t)$. Then, $x(t) = \sin t + A\sin(2t)$, and $v(t) = \cos t + 2A\cos(2t)$.

$$I = \frac{2S}{m} = \int_0^{2\pi} dt \left[4A^2\cos^2(2t) - A^2\sin^2(2t)\right] = 3\pi A^2 > 0.$$

Here again, we see that $\frac{dI}{dA} = 0$, as anticipated and required, but now the actual motion corresponds to the minimum of the action with respect to *this* variation. So, in this case, the actual motion is a saddle point. ✎

8.2 Characteristics of PoLA

PoLA = **Hamilton's principle**. The resulting equation of motion for L is called the **Lagrangian mechanics**, as opposed to the Newtonian mechanics.

1. It is a misnomer. Action is not always minimized. Rather, $\delta S = 0$ is all we mean and all we need. Minimum, maximum, or saddle point of the action functional—all of these are just fine, and do occur[5].

2. It deals with scalar quantities (action S and Lagrangian L) rather than vector quantities. This can be considered a welcome feature.

3. It is equivalent to, but more general than, the Newtonian formulation of classical mechanics. It applies almost everywhere in physics. When it does not, it is generalized easily (the Feynman path integral formalism of quantum mechanics).

4. It is mathematical. If a free body diagram (Newtonian mechanics) provides forces in full view, the Lagrangian mechanics appears to be just a bunch of mathematical equations, which may seem to "hide all physics under the rug." Newtonian mechanics forces you to think. Lagrangian mechanics may seem to force you to just do the math. Whether this is good or bad is up to you. One should always reflect upon the physics of the solution, though!

5. When bodies are in contact and the motions are constrained, the "force of constraint" (normal force, friction, ...) comes out of the mathematical formalism, "even if you don't try very hard." We will see how this works out in a later lecture.

6. The basic formalism applies only when there exists a potential function U, but the formalism can be extended when there are dissipative forces, using the concept of Raleigh dissipation function (cf. Landau or any other higher level mechanics text). We will not deal with such forces in the Lagrangian formalism in this course, except when they play the role of constraint forces. The fact that such a dissipative process must be put in by hand in addition to the above Hamilton's principle should not be viewed as weakening the case of this new formalism. As dissipative processes are not purely mechanical (they involve

[5]However, if t_1 and t_2 are different only by an infinitesimal time interval, then the minimum applies, for potential energies of common forms, e.g., the potential energy that depends on position only or the potential energy that depends on velocity only linearly (which is the case for particle experiencing Lorentz force). Namely, if $U(q, \dot{q}, t) = f(q,t) + g(q,t)\dot{q}$, then one can readily show that the action is minimized for an infinitesimal motion. The proof is quite similar to, while not identical with, what we did in Example 8.1.2.

heat and thus require statistical physics), this new formalism can be viewed as clearly separating mechanical matters and other matters.

8.3 Function and functional

There is no reason to feel weird about, or to fear, these words. A function can be viewed as a very large dimensional vector. Of course, a mathematical purist may like to say that a function is equivalent to an *infinite*, even worse *uncountably infinite*, dimensional vector.

For instance, for finite n dimensional real vectors,

$$|\vec{A}| = \sqrt{\sum_{i=1}^{n} A_i^2}, \qquad \text{magnitude} \tag{8.6}$$

$$\vec{A}\cdot\vec{B} = \sum_{i=1}^{n} A_i B_i. \qquad \text{inner product} \tag{8.7}$$

where A_i's and B_i's are the components of vectors $\vec{A}$ and $\vec{B}$.

It turns out real functions have very similar properties, and indeed the notion of "vector space" generalizes from finite dimensional vectors, with which any student of elementary Newtonian mechanics is familiar with, to any functions, real or complex. In order to see how this is done, you are advised to read mathematical physics books or mathematical books on functions.

In any case, for real functions, we can define[6]

$$\text{The magnitude of } f = \sqrt{\int_{-\infty}^{\infty} dx\, f(x)^2}, \tag{8.8}$$

$$\text{The inner product of } f \text{ and } g = \int_{-\infty}^{\infty} dx\, f(x)g(x). \tag{8.9}$$

Note that an integral is approximated as a sum, in any *actual* evaluation of it, and so one can view f as a finite dimensional vector with components, $f(x_i)/\sqrt{h}$, $i = 1, ..., N$. N is a very large number and $h = dx$ is a very small number; but, importantly, *both are finite in physics*. This is the sense in which any function can be considered as a very high dimensional vector. Lastly, note that in the above two equations, we do not say $f(x)$ or $g(x)$ on the left hand side; for instance, the magnitude of $f(x)$ means

[6] Here, we use the integration range as $(-\infty, \infty)$. In some cases, the physical space is constrained. If that is the case, then the integral should be adjusted to a semi-finite range or a finite range.

simply $|f(x)|$, the magnitude of a single component of the function, which is quite different from the magnitude of f, i.e., the *whole function* f.

So, then, what is a functional? It is a mapping that takes a function, i.e., a very large dimensional vector, and then returns a number. The magnitude of a function, as we defined above, is an example. The action, defined in Eq. 8.1, is another example.

Now, coming back to the functional derivative used in Eq. 8.5, we can now understand that it is merely a gradient in a very large dimensional vector space. So, the fact that the action is stationary means that the gradient of the action is zero.