Notes for Lecture 7

Driven SHO

In the previous lecture, we took care of differently damped oscillators pretty well. However, those were all free oscillators, in the sense that there were no driving forces. Namely, after the initial position and the initial velocity have been set, the oscillator was left alone and was allowed to follow whatever motion the characteristic parameters (natural frequency and damping) of the system dictates.

However, a more interesting situation arises when the oscillator is driven, i.e., when the oscillator is driven by an external force. The external force may be environmental (e.g., like vibrations on a bridge by wind) or some force exerted on the system on purpose (e.g., a beam of laser shining on a crystal, shaking all atoms in unison). In either case, it is the response of the system to the perturbing force that reveals the information about the nature of the system, and this is why the driven oscillator problem is such an important problem.

7.1 Particular solution, complementary function

Suppose that a force, $F(t)$, drives a SHO.

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{7.1}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \qquad f(t) \stackrel{def}{=} F(t)/m \tag{7.2}$$

$$Lx = f(t) \qquad L \stackrel{def}{=} \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2 \text{ is a } \textit{linear operator} \text{ acting on } x(t). \tag{7.3}$$

Now, we have a so-called **"inhomogeneous" differential equation**, or a **differential equation with a "source."** The $f(t)$ function is the source term.

No matter, we should remember that the general solution to this ordinary differential equation (ODE) is the one that has two integration constants (2 times 1 as the degree of freedom is 1). If we manage to find such a solution, then by the uniqueness of the solution of Newton's equation, that is *the* general solution.

How do we find them? The answer is that we already know a lot about the above equation in its homogeneous form ($f(t) = 0$). The general solution for the **homogeneous equation** is called a **complementary function**, $x_c(t)$. Note that the complementary function already contains two integration constants. What does this mean? **All we have to do is then to find one particular solution to the above equation.** Call that particular solution $x_p(t)$. This particular solution should not, and need not, have any integration constant[1].

Assume that we have found both $x_c(t)$ and $x_p(t)$. Then, the proof that $x(t) = x_c(t) + x_p(t)$ is the general solution is pretty simple, since L, as defined above, is a linear operator.

$$L(x_c + x_p) = Lx_c + Lx_p = 0 + f(t) = f(t). \qquad \text{QED.}$$

Let us just remind ourselves **what it means that L is a linear operator**. It means that for any numbers a, b and any functions $x_1(t), x_2(t)$,

$$L(ax_1(t) + bx_2(t)) = aLx_1(t) + bLx_2(t).$$

For the current SHO problem, $L = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$ is definitely a linear operator, due basically to the distributive rule of the differential operator $d(f+g)/dt = df/dt + dg/dt$.

A general approach that will give *any* particular solution x_p does exist (Green's function approach). Here, we use a more elementary approach. For either approach, though, the following central principle forms the foundation.

7.2 Superposition principle

A **linear system** displays this important principle. It means the following.

Suppose we have a system defined by the following equation

$$Lx(t) = f(t) \tag{7.4}$$

[1]Should you be writing the particular solution with some extra integration constants, that means you are repeating some part of the complimentary function in the particular solution. That is of course redundant, but there is no harm done.

where L is a **linear operator** on function $x(t)$, and $f(t)$ is a source term for this equation. Suppose that one can divide the source term into two terms, $f(t) = f_1(t) + f_2(t)$, and that

$$Lx_1(t) = f_1(t) \quad \text{and} \quad Lx_2(t) = f_2(t). \tag{7.5}$$

Then, the **superposition principle** means that $x_1(t) + x_2(t)$ is the solution for the original equation.

$$L(x_1(t) + x_2(t)) = f_1(t) + f_2(t) = f(t). \tag{7.6}$$

This is easy to prove since $L(x_1 + x_2) = Lx_1 + Lx_2$, due to the linearity of the operator L.

You will note that we made use of this linear property already, even though we did it without mentioning the name. For example, the solution for the homogeneous ODE was written as a linear combination of two independent solutions, and this was a trivial example of the superposition principle: it was the superposition principle with $f_1 = f_2 = 0$.

Notice that the superposition principle can be immediately extended to *any number of components* into which $f(t)$ can be decomposed: $f(t) = \sum_i f_i(t)$ or $f(t) = \int d\alpha \tilde{f}(\alpha, t)$. If each component gives rise to a certain solution ($f_i(t) \to x_i(t)$, $\tilde{f}(\alpha, t) \to \tilde{x}(\alpha, t)$), then the solution to the problem is

$$x(t) = \sum_i x_i(t), \qquad \text{for } f(t) = \sum_i f_i(t) \tag{7.7}$$

$$x(t) = \int d\alpha\, \tilde{x}(\alpha, t). \qquad \text{for } f(t) = \int d\alpha \tilde{f}(\alpha, t) \tag{7.8}$$

The functions $f_i(t)$ or $\tilde{f}(\alpha, t)$ can be any functions. Typical examples arise when we expand the function $f(t)$ in Fourier series[2], discrete for f_i and continuous for $\tilde{f}$. Physically the superposition principle means that each solution $x_i(t)$ or $\tilde{x}(\alpha, t)$ remains unchanged even when it is combined with all other solutions. Individual solutions "just add up." Note that, in this addition, what is added is the amplitude ($x_i(t)$ or $\tilde{x}(\alpha, t)$), not the intensity ($|x_i(t)|^2$ or $|\tilde{x}(\alpha, t)|^2$). This is the essential feature of the superposition principle, important for understanding the "interference phenomena" for light and other waves.

Indeed, the superposition property is an essential property of waves, as opposed to particles. Here, we are dealing with a SHO, which would not seem like much of a wave just yet. That is, it is not a traveling wave[3]. It can however be thought of as a

[2] Note that in this notation, f_i and $\tilde{f}$, if Fourier components, would include sinusoidal functions and multiplicative constants.

[3] If many SHOs are connected to each other, then we will have a traveling wave, as we will see later.

localized standing wave. We will deal more extensively with waves, near the end of this course.

As you can see, the superposition principle is a great property. It means that when an arbitrary form of a driving force $f(t)$ is given, then one can first decompose $f(t)$ to a sum (or an integral, if you like) of convenient components, solve the problem for each component force, and then sum (or integrate) all solutions!

Now, by the **Fourier theorem**, any "piecewise continuous" periodic function[4] can be written in a Fourier series[5]. Furthermore, any integrable function can be expressed as a Fourier integral. It then follows that if the response of a linear system to a force of a single sinusoid with an arbitrary angular frequency is known, then we know the response of the system to an arbitrary form of force[6].

So, this is the reason why we solve the driven SHO problem with a sinusoidal driving force in the next section.

Before we do that, let us come back to the physics point of view of the superposition. We mentioned that the principle of superposition means that each solution $x_i(t)$ (or $\tilde{x}(\alpha, t)$) "remains unchanged" when they are put together. Put another way, this means that if the principle of superposition breaks down then when those solutions are put together something new happens. Indeed, if a non-linear interaction is included, then some qualitative new behaviors occur.

7.3 Driven SHO, a sinusoidal force

So, consider a sinusoidal driving force $F(t) = F_0 \cos(\omega t)$, *without loss of generality*, as explained in the last section. Let $A = F_0/m$, and then the equation to solve becomes:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos(\omega t). \tag{7.9}$$

Here, we will assume[7]

$$A = \frac{F_0}{m} > 0. \tag{7.10}$$

[4] "Piecewise continuous" means that except at a finite number of points in any arbitrary finite interval, the function is continuous.

[5] If you are not familiar with the Fourier theorem, it is time to revisit it in your mathematical physics book.

[6] Assuming that we can do the Fourier transform and the inverse-Fourier transform.

[7] If F_0 is negative, then we can shift the origin of time so that we can make A positive: $A\cos(\omega t + \pi) = -A\cos(\omega)$.

. . . Let us go to the complex world (I mean the complex plane)[8].
The above equation turns into

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x \;=\; A\exp(i\omega t). \tag{7.11}$$

We solve for a *complex* $x(t)$ and then, at the end, can take the real part of it, as the actual EOM is the real part of the complex equation that we just set up. Assume

$$x(t) \;=\; C\exp(i\omega t). \tag{7.12}$$

Plugging this into the equation of motion, and we get

$$C(-\omega^2 + 2\beta\omega i + \omega_0^2)\exp(i\omega t) \;=\; A\exp(i\omega t). \tag{7.13}$$

Since this equation should hold at any time, we must conclude

$$C(-\omega^2 + 2\beta\omega i + \omega_0^2) \;=\; A, \tag{7.14}$$

$$C \;=\; \frac{A}{\omega_0^2 - \omega^2 + 2\beta\omega i}. \tag{7.15}$$

Let us define

$$D \overset{def}{=\!=} |C|. \tag{7.16}$$

Then, we get

$$D \;=\; \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}. \tag{7.17}$$

In considering the phase of C, note that C can be rewritten as

$$C \;=\; \frac{A(\omega_0^2 - \omega^2 - 2\beta\omega i)}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}. \tag{7.18}$$

We shall assume $\omega_0 > 0$ and $\beta > 0$ in the following discussion. By convention, $\omega \geq 0$. When $\omega = 0$, we see that C is on the positive real axis. As ω increases from 0, C

[8] As you might have noticed, from the previous lecture, we have been considering complex numbers in a very relevant manner. Indeed, complex numbers are very convenient if we like to solve SHO problems easily. We do *not* need to use any complex numbers in this problem, or any other classical mechanics problem, but, life becomes extraordinarily easier, if we do for certain problems like SHO problems or waves.

goes into the 4th quadrant, since the imaginary part is negative while the real part remains positive. As ω increases further, its real value vanishes at $\omega = \omega_0$. At that point, C crosses over to the 3rd quadrant. As $\omega \to \infty$, it remains in the 3rd quadrant. Considering that C remain in the lower half plane or on the real axis, it makes sense to define its phase as $-\delta$. It is given by, from the above equation of C,

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right), \qquad \delta \geq 0. \tag{7.19}$$

Then,

$$x(t) = D\exp(-i\delta)\exp(i\omega t). \tag{7.20}$$

. . . Now, we come back to the real world (I mean the real axis).
Taking the real part, we get

$$x(t) = D\cos(\omega t - \delta), \tag{7.21}$$

with D and δ as given above. δ is the "**phase shift/lag**."

Before we go on further, let us note that the full solution is of the form

$$x_c(t) + x_p(t)$$

What we obtained just now is $x_p(t)$. We already know what $x_c(t)$ is from the previous lecture (the solution discussed in the damped SHO section). When a finite damping is present, $x_c(t)$ is always damped with the damping constant given by β (under-damping) or $\beta - \sqrt{\beta^2 - \omega_0^2}$ (over-damping[9]). This means that if we wait for time $\gg 1/\text{damping constant}$, $x_c(t)$ is negligibly small. The solution that applies at such a large value of time is called the **steady state solution**. The solution that applies to the initial time when both $x_c(t)$ and $x_p(t)$ are appreciable is called the **transient solution.** Here, we will discuss the steady state solution only: $x_p(t)$.

[9] Here, we choose the slower damping of the two damping constants possible, $\beta \pm \sqrt{\beta^2 - \omega_0^2}$, in view of the subsequent discussion.

Let us analyze the phase lag/shift function $\delta(\omega)$ a bit. Using $\tan z \approx z$ for $|z| \ll 1$, we get

$$\delta \approx 2\beta\omega/\omega_0^2, \qquad \text{when } \omega \approx 0. \tag{7.22}$$

So, near $\omega = 0$, $\delta(\omega)$ has a positive slope, and the slope increases as damping increases.

The more important case to examine is the case when $\omega \approx \omega_0$. By defining

$$\eta \stackrel{def}{=\!=} \frac{\omega - \omega_0}{\omega_0}, \tag{7.23}$$

we see that $\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx -2\omega_0^2\eta$. And, $\omega = \omega_0(1+\eta)$. Therefore, we get

$$\frac{2\beta\omega}{\omega_0^2 - \omega^2} \approx \frac{2\beta\omega_0(1+\eta)}{-2\omega_0^2\eta} \approx -\frac{2\beta}{\omega_0\eta}. \tag{7.24}$$

The inverse tangent of this quantity is δ: i.e., $\tan\delta \approx -\frac{2\beta}{\omega_0\eta}$. Using $\tan\delta \approx \tan(\pi/2 + \tilde{\delta}) \approx -\cot(\tilde{\delta}) \approx -1/\tilde{\delta}$, where $\tilde{\delta} \stackrel{def}{=\!=} \delta - \pi/2$, we get

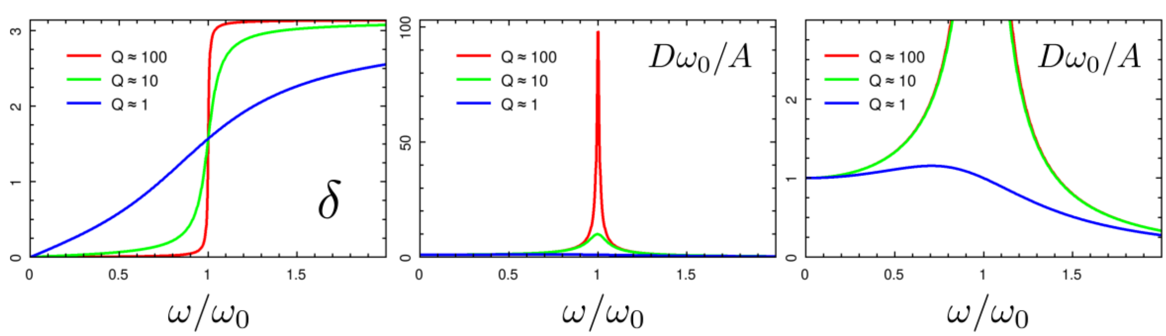
$$\delta \approx \frac{\pi}{2} + \frac{\omega_0\eta}{2\beta} \tag{7.25}$$

$$= \frac{\pi}{2} + \frac{\omega - \omega_0}{2\beta}. \qquad \text{for } \omega \approx \omega_0 \tag{7.26}$$

This is the **resonance behavior** of the phase shift. As ω is tuned from just below ω_0 to just above ω_0, δ goes through $\pi/2$ from below. $\delta = \pi/2$ at $\omega = \omega_0$, and the slope there goes to infinity as $\beta \to 0$. When $\beta \to 0$, $\delta = 0$ if $\omega < \omega_0$ and π if $\omega > \omega_0$, and so this infinite slope is reasonable. Finally, if $\omega \to \infty$, we get

$$\delta \approx \tan^{-1}\frac{2\beta}{-\omega} \approx \pi - \frac{2\beta}{\omega}, \qquad \omega \to \infty. \tag{7.27}$$

Here are some plots that demonstrate the behaviors that we just discussed.



In these plots, the three colors correspond to the three different values of β: $\beta = 0.005\omega_0$, $0.05\omega_0$, and $0.5\omega_0$. Anticipating the Q factor (Eq. 7.30), we have represented these three values of β as $Q \approx 100$, $Q \approx 10$, and $Q \approx 1$, respectively. In the δ plot, we plotted Eq. 7.19 as a function of dimensionless frequency ω/ω_0. And in the $D\omega_0/A$ plots (the two plots on the right, the same plot shown in different y ranges) we used Eq. 7.17 to plot $D\omega_0/A$, a dimensionless form of D, as a function of ω/ω_0.

How do we understand $\pi/2$ phase lag at $\omega = \omega_0$? If you are good at pumping a swing, convince yourself[10] that this is consistent, that is, you apply torque exactly the quarter cycle before the amplitude becomes maximum. Note that when $\delta = \pi/2$, the power is delivered to the system by the external force in the most optimum way, since $x = D\cos(\omega_0 t - \pi/2) = D\sin(\omega_0 t)$ and thus $v = D\omega_0\cos(\omega_0 t)$. The power = Fv, and F and v are exactly in phase, both behaving as $\cos(\omega_0 t)$, when $\omega = \omega_0$.

Let us analyze the function $D(\omega)$ a bit as well. It is recognized immediately that if $\beta = 0$, then $D \to \infty$ at $\omega = \omega_0$. Thus, D in general is expected to have a peak structure around the natural frequency of the system. This is the so-called **resonance** behavior. Near $\omega \approx 0$, $D \approx A/\omega_0$. The maximum of D is obtained by putting $dD/d\omega = 0$, which means $2(\omega^2 - \omega_0^2)2\omega + 8\beta^2\omega = 0$, which means $\omega = \sqrt{\omega_0^2 - 2\beta^2}$. This is the so-called **amplitude resonance frequency**, ω_R.

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}. \tag{7.28}$$

Notice that $\omega_R < \omega_1 = \sqrt{\omega_0^2 - \beta^2} < \omega_0$, where ω_1 is the oscillation frequency with under-damping.

If β is small, then $\omega_R \approx \omega_0$. Near the peak, consider the *intensity* profile:

$$D^2 \approx \frac{A^2}{(\omega^2 - \omega_R^2)^2 + (2\beta\omega_R)^2} \approx \frac{A^2}{(2\omega_R)^2[(\omega - \omega_R)^2 + \beta^2]}. \tag{7.29}$$

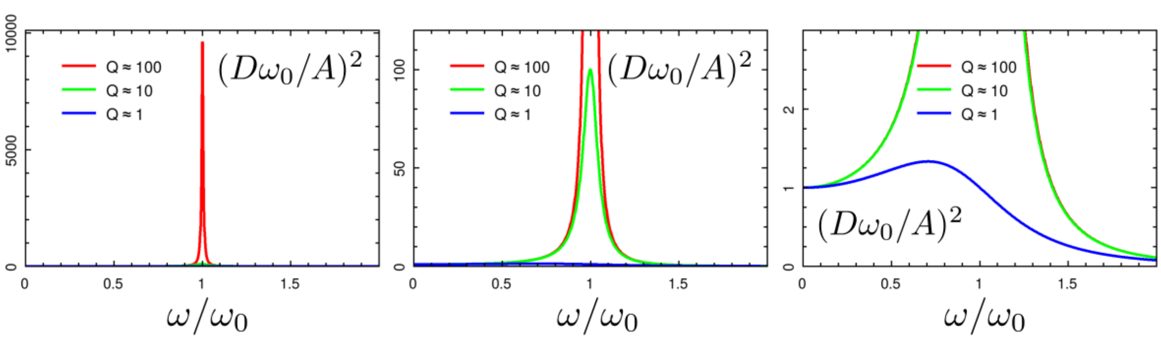
This is the so-called Lorentzian line shape, centered at ω_R with the full width at half maxima (FWHM), 2β. For this reason, it is customary to define the Q **factor ("quality factor")**:

[10] Or, not?? Pumping a swing is not an entirely obvious physics problem, according to some authors.

$$Q \stackrel{def}{=} \frac{\omega_R}{2\beta}. \tag{7.30}$$

As such, Q defines how sharp the resonance behavior is, as ω is swept. Poor quality oscillator with a large damping has a poor resonance characteristics. For instance, for a critically damped or a over-damped oscillator, the resonance characteristics will be very poor. High Q factor is required for precise measurements.

Here are three panels of plots. The three panels contain the same graphs, which are shown in different y ranges in the three panels. Here, a dimensionless form of D^2 is plotted by squaring Eq. 7.17 for $\beta = 0.005\omega_0$ ($Q \approx 100$), $\beta = 0.05\omega_0$ ($Q \approx 10$), and $\beta = 0.5\omega_0$ ($Q \approx 1$). Observe that for high Q values, the resonance is very sharp. For small Q value (≈ 1), the resonance is very weak. Also note that, in this dimensionless form the maximum of $(D\omega_0/A)^2$ is given by $\approx Q^2$, which can be verified easily from Eq. 7.17. Finally, the shapes shown by the case $Q \approx 10$ or $Q \approx 100$ exemplifies a Lorentzian shape[11].



The resonance frequencies for the amplitude and the potential energy are identical, since the potential energy is proportional to the amplitude squared. However, the kinetic energy of the SHO is not necessarily maximized at the same frequency. As

$$x \propto \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\omega^2\beta^2}} \cos(\omega t - \delta), \tag{7.31}$$

[11]The Lorentzian function tends to the Dirac delta function as it becomes infinitely sharp. More specifically, in the current case, as $Q \to \infty$, $(D\omega_0/A)^2 \to \frac{\pi}{2} Q \delta\left(\frac{\omega}{\omega_0} - 1\right)$, where δ is the Dirac delta function.

it follows that

$$\dot{x} \propto \frac{-\omega}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\omega^2\beta^2}} \sin(\omega t - \delta), \tag{7.32}$$

$$K \propto \dot{x}^2 \propto \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\omega^2\beta^2} \sin^2(\omega t - \delta). \tag{7.33}$$

where K is the kinetic energy[12]. Let us use **the following *very important* result**

$$\frac{1}{T}\int_{t_0}^{t_0+T} dt \sin^2(\omega t - \delta) = \frac{1}{2}, \qquad \text{where } T \stackrel{def}{=} \frac{2\pi}{\omega} \text{ and } t_0 \text{ is any real number.} \tag{7.34}$$

Then, the average kinetic energy $\langle K \rangle$ over the period T is given by

$$\langle K \rangle \propto \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\omega^2\beta^2} \cdot \frac{1}{2}. \tag{7.35}$$

Taking the ω-derivative, $d\langle K \rangle/d\omega \propto \omega[(\omega^2 - \omega_0^2)^2 + 4\omega^2\beta^2] - \omega^2[(\omega^2 - \omega_0^2)2\omega + 4\omega\beta^2] = \omega(-\omega^4 + \omega_0^4)$, where we have omitted the common denominator, not important for finding zeros of $d\langle K \rangle/d\omega$. So, $\langle K \rangle$ has a maximum[13] at $\omega_E = \omega_0$. This would be the **kinetic energy resonance frequency**, occurring exactly at the natural frequency of the system, different from the amplitude/potential-energy resonance frequency $= \omega_R = \sqrt{\omega_0^2 - 2\beta^2}$.

7.4 Kirchoff law and SHO

Let us consider a circuit with a battery suppling an emf E and three circuit elements, an inductor, a capacitor, and a resistor, in series, forming a closed circuit. There is an exact analogy between the circuit and the mechanical system.

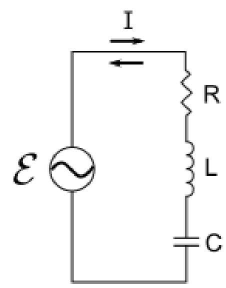
$$L\frac{dI}{dt} + \frac{Q}{C} + RI = \mathcal{E}. \tag{7.36}$$

[12] The popular symbol for the kinetic energy is T or K. Here, we better use K, since we use T for the period.

[13] Here, we just showed that $\langle T \rangle$ has a unique extremum at $\omega = \omega_E$ for $\omega >= 0$. How do we know it is the maximum? If you are not sure, do NOT take the 2nd derivative. That would be too complicated. Rather, note that $\langle T \rangle \to 0$ when $\omega \to 0$ or ∞, while $\langle T \rangle$ = finite at $\omega_E = \omega_0$. Since ω_E is the only extremum point between $\omega = 0$ and $\omega = \infty$, it must be a maximum.

Noting that $I = \dot{Q}$, the circuit equation becomes

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \mathcal{E}. \tag{7.37}$$



The following correspondence can then be noted, if we compare this equation with Eq. 7.1.

Circuit	Mechanical
Q	x
L	m
R	b
$1/C$	k
$\mathcal{E}$	F

The current problem becomes an RLC resonant circuit problem, if driven by an emf $\mathcal{E} = E_0 \cos \omega t$. Let us ask ourselves what is the resonance frequency to maximize V_L, the voltage on the inductor. The voltage $V_L = L\ddot{Q}$. So, this is equivalent to maximizing the acceleration. We can use our solutions for the mechanical system, if we make the following change of symbols.

$$\omega_0^2 = k/m \to 1/(LC),$$

$$\beta = b/(2m) \to R/(2L),$$

$$A = F_0/m \to E_0/L.$$

The result is

$$Q = \frac{E_0}{\sqrt{(L\omega^2 - 1/C)^2 + R^2\omega^2}} \cos(\omega t - \delta) \tag{7.38}$$

$$I = \frac{-E_0 \sin(\omega t - \delta)}{\sqrt{(L\omega - 1/(C\omega))^2 + R^2}} \tag{7.39}$$

$$V_L = L\frac{dI}{dt} = \frac{-E_0\omega \cos(\omega t - \delta)}{\sqrt{(L\omega - 1/(C\omega))^2 + R^2}} \tag{7.40}$$

$$\stackrel{def}{=} V(\omega)\cos(\omega t - \delta) \tag{7.41}$$

To maximize $|V(\omega)|$, note that $d|V(\omega)|^2/d\omega \propto 2\omega((L\omega - 1/(C\omega))^2 + R^2) - \omega^2 2(L\omega - 1/(C\omega))(L + 1/(C\omega^2)) \propto (LC\omega^2 - 1)^2 + R^2C^2\omega^2 - (LC\omega^2 - 1)(LC\omega^2 + 1) \propto (R^2C^2 - 2LC)\omega^2 + 2$, again, ignoring common denominators since all we are going for is the zero of $d|V(\omega)|^2/d\omega$. Setting this expression to 0, we get

$$\omega_{max} = \frac{1}{\sqrt{LC - \frac{R^2C^2}{2}}}. \tag{7.42}$$

This is the frequency at which the acceleration is maximized. It is different from $\omega_E = \omega_0 = 1/\sqrt{LC}$ and the amplitude resonant frequency $\omega_R = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{1/LC - R^2/(2L^2)}$.

7.5 Quality factor

Discussions in this section go a bit deeper into the meaning of all we have discussed so far, and they are not required for your reading. They can be read later, should they seem confusing. Also, while some quantum mechanics is discussed here, quantum mechanics should not be considered in any other parts of this course, for the clarity's sake (unless we make a small detour like this).

What is the physical meaning of the quality factor (Eq. 7.30)? We already discussed it very briefly after that definition, but perhaps more discussion is necessary in order to really understand what "quality" means here.

In short, the quality means the quality of the SHO system involved. If the quality of the oscillator is high, then it will give a precise and long-lasting response. If the quality of the oscillator is low, then it will give a mucky and short-lived response. Perhaps you can compare the quality of the sound that is generated with professional equipment[14] to the quality of the sound that is generated with make-shift equipment

[14] For instance, a guitar made by Brian May.

such as a partially filled bottle or a simple string stretched across the opening of a cup. A professional instrument produces a tone that is clear and long lasting when we play it. We play it by applying some sort of force, and the instrument responds, and so this is a driven oscillator problem. Long lasting means that β is small, since if we pluck a guitar, for example, and stop, then the sound amplitude will fade away with the time constant $1/\beta$. On the other hand, for a make-shift instrument, the sound typically lasts for a short time, since β is large. Also, observe that a make-shift instrument produces an unclear tone—the resonance response is very weak, and so the tone generated by the instrument has a wide range of frequencies involved. The sound is unclear in its tone, although we may recognize the average tone as a certain note.

Perhaps you can recognize a certain relation between the lifetime of the "wave" with the uncertainty of the frequency of the "wave"[15]. Indeed, in the quantum mechanical sense, what we are discussing here is what is called the Heisenberg uncertainty principle for time and energy[16]. We shall limit our discussion here to the slightly underdamped oscillator case $\beta \ll \omega_0$, although the discussion can be generalized to all other cases. The time scale in which the amplitude dies away is given by $1/\beta$. The "uncertainty" in energy is in this case directly proportional to the uncertainty in frequency, which is given by $\sim 2\beta$.

Now, why is the uncertainty in frequency $\sim 2\beta$? This becomes clear if you just consider the response of the system D^2 that we plotted above. Suppose that you are examining a sample system and you like to figure out what the natural frequency, or the resonant frequency, of that system is. What you must do is then to couple the system to an external perturbation (e.g., a tunable laser light source, if the relevant frequency is in the domain of laser lights) and measure the response of the system. The response of the system (absorption, emission, or transmission) will all carry the signature of the resonant profile that we calculated above[17] for D^2. *It is from such response that we measure and define the natural frequency of the system.* If the response is very sharp, then it means that the system is clean with a sharp natural frequency. If the response is very broad, then it means that the natural frequency is very ill-defined, since the system apparently responds to very many frequencies (the system is "not very clean"). Since the D^2 profile has the width $\sim 2\beta$, when $\beta \ll \omega_0$, we see that this quantity (2β) *defines the uncertainty of the natural frequency* of the system, in this very rigorous physical way[18].

[15] We can start discussing waves here, since waves are simply a coordinated bunch of SHO motions.

[16] Recall that $\hbar\omega$ is an energy scale.

[17] This is of course assuming that our simple-looking theory is relevant for a real system. This is quite more likely the case than not, if a real system is a reasonably linear system and not too much damping is present.

[18] In previous sections, we defined ω_R in a very precise way. That precise definition has a meaning only in an "average" sense, as the actual response of the system occurs in a spectrum of frequencies,

So, if you like to peek into quantum mechanics, just temporarily imagining that the oscillator that we defined so far is a quantum one, then we conclude that $\Delta\omega\Delta t \sim 2\beta\frac{1}{\beta} \sim 2$, where $\Delta\omega$ is the uncertainty in frequency and Δt is the lifetime of the oscillator. By multiplying $\hbar$, we get $\Delta E\Delta t \sim 2\hbar$, which is, up to an unimportant numerical factor, the core content of the Heisenberg uncertainty principle for energy and time: $\Delta E\Delta t \sim \hbar$.

So much for quantum mechanics, but the same argument for $\Delta\omega$ and Δt is valid for all physics or engineering problems involving oscillators and waves, whether or not the relevant oscillator is big (classical) or (sub-)nano-scale small (quantum). In general physics or engineering measurements, an oscillator with high Q can act as a very good reference for timing measurements, since the uncertainty in frequency, $\Delta\omega$, or the quality factor to be more precise, is what determines the intrinsic error bar (i.e., the best possible resolution) of the timing measurement. Here is how you can see it. Since the period $T = \frac{2\pi}{\omega}$, we see that the relative uncertainty of our measurement of time is limited by[19]

$$\frac{\Delta T}{T} = \frac{\Delta\omega}{\omega} \tag{7.43}$$

where $\Delta\omega$ is the intrinsic uncertainty of the natural frequency of our oscillator (i.e., here it must be understood that $\omega = \omega_R$ and $\Delta\omega = \Delta\omega_R$ or $\omega = \omega_0$ and $\Delta\omega = \Delta\omega_0$). Notice that the intrinsic timing resolution (ΔT) is directly proportional to the inverse of the quality factor, Q, for small under-damping case, since $\Delta\omega_R = 2\beta$, when measured as the FWHM. So, the higher the quality factor, the better the timing resolution. Another way of putting it is that the Q factor ($\omega_R/\Delta\omega_R$) is equal to the resolving power ($T/\Delta T$) of the timing measurement.

not just at a single frequency.

[19] Here, both ΔT and $\Delta\omega$ are defined as positive.