Notes for Lecture 6

Small Oscillations

We consider small oscillations around a stable equilibrium point. We assume that the second derivative of the potential is non-zero. The resulting motion is a simple harmonic motion (SHM), an extremely important kind of motion. It is a ubiquitous phenomenon for stable physical systems.

6.1 Free simple harmonic oscillator (SHO)

Let us consider a simple one dimensional SHO, i.e., a simple mass on spring problem. Assuming small deformation of the spring, the Hooke's law potential and force apply.

$$U(x) = \frac{1}{2}kx^2, \tag{6.1}$$

$$F(x) = -kx. \tag{6.2}$$

Note that this potential energy has no upper limit, and so we can have only bound motions. Any bound motion in 1D with conservative forces alone is a periodic motion, as discussed in the last lecture.

Let us write down the equation of motion and its parameter.

$$m\ddot{x} = -kx \qquad \text{mass on spring} \tag{6.3}$$

$$\ddot{x} = -\omega^2 x \qquad \text{SHO equation of motion} \tag{6.4}$$

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{angular frequency} \tag{6.5}$$

As shown above, the SHO equation of motion contains a single parameter, ω, which is the **angular frequency**. It is often referred to as the **natural (angular) frequency**. $\frac{\omega}{2\pi}$ is referred to as the **frequency** (ν or f is a symbol frequently used for this). And the inverse of ν is the **period**, which is often referred to as T: $T = \frac{1}{\nu} = \frac{2\pi}{\omega}$. Note that some physicists use ω for most of their work and some ν. In physics, it is arguably easier to use ω, instead of ν. So, in this course, we will avoid using ν or f whenever it is possible, and we will use ω to refer to the frequency of the system.

So, when the word "frequency" is used in this course, we will mean "angular frequency" automatically, unless specified otherwise in rare cases.

For the natural frequency it is customary to add subscript 0 or N, and so the SHO equation of motion is given by

$$\ddot{x} = -\omega_0^2 x. \qquad \text{simple harmonic equation of motion} \qquad (6.6)$$

6.1.1 Conserved quantities

According to the conservation principles that we summarized in the last lecture, here we have only one conserved quantity: the energy is conserved, since any force derivable from a potential is a conservative force.

The momentum is not conserved, since we have non-zero $F(x)$.

Now, given the potential energy $U(x)$ of Eq. 6.1, the total mechanical energy, $E \geq U_{min} = 0$. Therefore, the energy E is conserved and it is non-negative.

6.1.2 Solutions

How about the general solution to the equation of motion, Eq. 6.6?

It is available in three different forms, and you can pick whichever form that you find convenient. But, if you choose one form, then you cannot mix it without another form, unless you re-define some repeating symbols (A or ϕ_0; see box below).

$$x(t) = A\cos(\omega_0 t + \phi_0) \qquad (6.7)$$
$$x(t) = A\sin(\omega_0 t + \phi_0) \qquad (6.8)$$
$$x(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) \qquad (6.9)$$

Any of these three forms of solutions is good for Eq. 6.6, as can be verified by finding $\ddot{x}(t)$ and seeing that it is equal to $-\omega_0^2 x(t)$. This verification work if left for your exercise. As each form contains two integration constants (A, ϕ_0 or A, B), we have the general solution to the SHO problem.

Symbols are confusing.

Physicists can't live without symbols. Sometimes it would appear that they can't live with them, since, in almost every physics book, you will find the same symbol is used for different meanings (hopefully in different chapters). Unfortunately, that is life. Most of the time, the context is clear, and there is not a problem. You must get used to "reading the context." For instance, in the above solutions, the three A's are *not* necessarily the same A, and neither are the two ϕ_0's.

If you like challenge or if you are still confused, then here is an exercise for you. Rewrite the second form and the third from above as $B\sin(\omega_0 t + \phi_1)$ and $C\sin(\omega_0 t) + D\cos(\omega_0 t)$, and find each of B, ϕ_1, C, and D as a function of A and ϕ_0, by equating one form of solution to any other form.

While the above way of obtaining solutions is just fine, we can also demonstrate another method in this case. We can use the equation that we derived in the last lecture for any 1D problem with potential energy.

$$t = \pm \int_{x_0}^{x} dx' \sqrt{\frac{m}{2(E - U(x'))}} \qquad x_0 \equiv x(t=0) \tag{6.10}$$

where $U(x)$ is any potential energy and one of the sign $\pm$ can be chosen for convenience. However, only one sign must be chosen.

Here, we choose the minus sign, since it is slightly more convenient.

$$\begin{aligned}
t &= -\int dx \sqrt{\frac{m}{2(E - \frac{1}{2}kx^2)}} \\
&= -\frac{1}{\omega_0}\int dx \frac{1}{\sqrt{A^2 - x^2}} && \omega_0 \stackrel{def}{=\!=} \sqrt{\frac{k}{m}},\ A \stackrel{def}{=\!=} \sqrt{\frac{2E}{k}} \\
&= \frac{1}{\omega_0}(\cos^{-1}(x/A) - \cos^{-1}(x_0/A)) \\
&= \frac{1}{\omega_0}(\cos^{-1}(x/A) - \phi_0) && \phi_0 \stackrel{def}{=\!=} \cos^{-1}(x_0/A)
\end{aligned}$$

Therefore, the solution is

$$x(t) = A\cos(\omega_0 t + \phi_0)$$

which is precisely the first form of solution that we listed above.

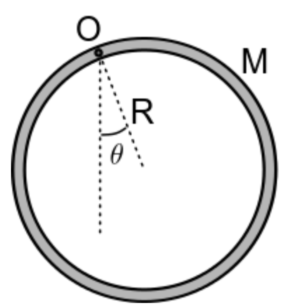
With this form of solution (or the second form), we define the following terms.

Amplitude A can always be taken as positive, since if A is negative, then $\phi_0 \to \phi_0 + \pi$ will make it positive. From now on then, we will assume, without loss of generality, that $A \geq 0$. This A is referred to as the **amplitude**. This is "amplitude" in a narrow sense of the word. In a broad sense of the word, sometimes $x(t)$ itself is referred to as amplitude.

Phase $\phi = \omega_0 t + \phi_0$ is defined as the phase. ϕ_0 is the initial phase.

6.1.3 Example—a physical pendulum

A ring shaped object is hanging from a pivot point (O), which is assumed to provide a frictionless support for the ring. The ring is oscillating back and forth, while the ring remains in a fixed plane (i.e., no motion perpendicular to the ring plane). Assuming that the radius of the object is R and its mass is M, what is the frequency of the oscillation for oscillation with a small amplitude? Ignore the thickness of the ring.



If the ring were rotating around its center, then the rotational inertia is given by MR^2. The rotational inertia around O is given by this value plus MR^2 by the parallel axis theorem, and is given by $2MR^2$.

The rotational motion has the same form of equation of motion as

$$I\ddot{\theta} = -\kappa\theta \tag{6.11}$$

if the potential energy is given by $U(\theta) = \frac{1}{2}\kappa\theta^2$. The potential energy of this problem is given by

$$U = Mgy = Mg(R - R\cos\theta) \approx \frac{1}{2}MgR\theta^2, \tag{6.12}$$

if the height of the center of the ring, y, is measured relative to the center of the ring when the ring is at the lowest point of motion.

Thus, in this case, $\kappa = MgR$. Combining this with $I = 2MR^2$, we get the answer.

$$\omega_0 = \sqrt{\frac{\kappa}{I}} = \sqrt{\frac{g}{2R}}. \tag{6.13}$$

Generalization

Let us consider a simple pendulum with mass M and the distance from the mass to the pivot point R. We know that it has the frequency

$$\omega_{0,simple-pendulum} = \sqrt{\frac{g}{R}}.$$

Is it a coincidence that this value is greater than the above physical pendulum with the same distance from pivot to the center of mass? Of course, not. In general, the rotational inertia of the physical pendulum around its center of mass is given by

$$I_{cm} = \gamma MR^2, \qquad \gamma \geq 0, \tag{6.14}$$

where γ is a dimensionless number, dependent on the shape and the mass distribution of the object (0 for point mass and positive for any extended object). So, the rotational inertia to use for a physical pendulum is, again using the parallel axis theorem,

$$I = I_{cm} + MR^2 = (1+\gamma)MR^2. \tag{6.15}$$

This means that a physical pendulum has the following general expression for the frequency

$$\omega_0 = \sqrt{\frac{g}{(1+\gamma)R}}. \tag{6.16}$$

6.2 Phase space

As we will see, the concept of the **phase space** plays an important role in physics. Consider a system with M degrees of freedom. For N particles in D dimensions, $M = ND$. Then, we have M spatial coordinates, D per each particle: x_1, x_2, ... x_M. Now, consider a $2M$ dimensional space, where, for each x_i axis, we add an axis for p_i, where p_i is the corresponding momentum. This space is referred to as the phase space.

Why is this space called the phase space? From the simple harmonic motion point of view, it makes a lot of sense. Let us see why.

Note that the energy is conserved and its value is given by $E = \frac{1}{2}kA^2$. The energy conservation equation is given by

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2. \tag{6.17}$$

Multiplying this equation by $\frac{2}{k}$, and using $\frac{m}{k} = \frac{1}{\omega_0^2}$, we get

$$x^2 + \left(\frac{v}{\omega_0}\right)^2 = A^2. \tag{6.18}$$

This means that if we define the phase space[1] as the (X, Y) space where $X = x$ and $Y = -\frac{v}{\omega_0}$, then the simple harmonic motion is represented as a uniform circular motion in the (X, Y) space.

This is no coincidence at all. I leave it as your exercise to prove that

$$\ddot{X} = -\omega_0 \dot{Y} \tag{6.19}$$

$$\ddot{Y} = \omega_0 \dot{X} \tag{6.20}$$

which have exactly the same form as Eqs. 4.7 and 4.8.

In a circular motion context, "phase" means "angle," which is precisely the meaning of $\phi = \omega_0 t + \phi_0$ in the current problem.

Keep in mind the following properties of the phase space.

- Specifying the initial condition of a system is equivalent to specifying a point in the phase space at time 0.

[1] You may protest that this is not exactly as we defined above. Namely, Y is not exactly p_x, the momentum for x. This does not raise any issue, if we allow the flexibility for the scale (or the unit of measurement) and the sign, since Y = a constant times mv.

- The time evolution of that point is completely determined by Newton's laws.
- The motion of a particle, or a system of particles, is represented as a single path in the phase space.
- Suppose you prepare an otherwise identical system with two different initial conditions. As time goes on, the two points will move in phase space. Those two points can never occupy the same point in phase space at the same time.
- Such two paths either are identical (if the two initial conditions are related to each other by a time offset), or never intersect each other.

Phase space has the fundamental importance in physics. It is also important for engineering as well—an important theory to consider when building electron guns is based in the phase space.

6.3 Damped SHO

A more realistic oscillator has damping/friction. Let us model it as a frictional force $-bv$ ($b > 0$). Then, the equation of motion to solve becomes

$$m\ddot{x} = -kx - bv, \tag{6.21}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \qquad \beta \overset{def}{=\!=} b/(2m). \tag{6.22}$$

Here, β is called the **damping parameter**.

Damping means no energy conservation, and so we cannot use the method that we used for the free SHO, in this case.

What is the general method to solve this important equation?

Note that the EOM is linear. What does the linearity mean? Define

$$L \overset{def}{=\!=} \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2. \tag{6.23}$$

L is an operator acting on $x(t)$. The EOM can be written as

$$Lx(t) = 0$$

Due to L being a *linear*[2]operator, if we manage to find two independent solutions, $x_1(t)$ and $x_2(t)$ to $Lx(t) = 0$, then we can form a linear combination with two constants, $A_1x_1 + A_2x_2$, which would be the general solution to the problem, since it

[2] $L(A_1x_1 + A_2x_2) = A_1Lx_1 + A_2Lx_2$ for any numbers A_1, A_2 and functions x_1, x_2

contains the correct number of integration constants (A_1 and A_2). The art is how to find x_1 and x_2. While a more systematic theory such as the Sturm-Liouville theory or a series expansion method exists, the current problem is a well-known problem and one has to be very familiar with how to obtain solutions easily. *Here it goes!*

We try $x(t) = \exp(\alpha t)$. We get

$$(\alpha^2 + 2\beta\alpha + \omega_0^2)e^{\alpha t} = 0. \tag{6.24}$$

In order for this to hold for any t value, we must require that

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0, \tag{6.25}$$

which means

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \tag{6.26}$$

So, the general solution is

$$x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}\, t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}\, t\right)\right], \tag{6.27}$$

if $\omega_0^2 \neq \beta^2$. This solution describes both under-damping ($\beta^2 < \omega_0^2$) or over-damping ($\beta^2 > \omega_0^2$).

What if $\omega_0^2 = \beta^2$ (critical damping)? Then, there is only one α value ($-\beta$), and so the above solution reduces to one function ($\exp(-\beta x)$) only. So, is there only one solution? Newton's law says no (Lecture 2)! There *must* be another solution. In this case, by direct substitution, $te^{-\beta t}$ is shown to be a solution, and so the general solution is (for $\omega_0^2 = \beta^2$):

$$x(t) = e^{-\beta t}(A_1 + A_2\, t), \qquad \text{critical damping.} \tag{6.28}$$

How could we have guessed that the other solution is $te^{-\beta t}$? You can take a more systematic point of view as follows. We write down the solution as $g(t)e^{-\beta t}$ and then figure out what kind of equation $g(t)$ must satisfy: it is $\ddot{g}(t) = 0$ (left as exercise).

6.3.1 Under-damped SHO

For an under-damped SHO ($\omega_0 > \beta$), we define

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}. \tag{6.29}$$

Then,

$$x(t) = e^{-\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]. \tag{6.30}$$

Well, why did we get a complex quantity here? The answer is that the above EOM is perfectly happy with a complex solution. We are not necessarily unhappy with it, but should know that the physical solution must be real. We must require that the solution is a real number. That means $x^*(t) = e^{-\beta t}\left[A_1^* e^{-i\omega_1 t} + A_2^* e^{i\omega_1 t}\right] = x(t)$. This is possible only if $A_1 = A_2^*$. Write $A_1 = Ae^{i\phi_0}/2$, where $A > 0$. Then,

$$x(t) = e^{-\beta t} A\left[e^{i(\omega_1 t+\phi_0)} + e^{-i(\omega_1 t+\phi_0)}\right]/2. \tag{6.31}$$

Thus, we get

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t + \phi_0). \tag{6.32}$$

- If β becomes small, then the solution approaches the correct limit (free SHO), since $\beta \to 0$ and $\omega_1 \to \omega_0$:

  $$x(t) = A\cos(\omega_0 t + \phi_0)$$

  In the current context, it means that the general solution for

  $$\ddot{x} = -\omega_0^2 x$$

  can be written as

  $$x(t) = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t}$$

  This **complex solution** for the free SHO is worth remembering.

- If β is small, the exponential decay term will decay very slowly, while the cosine term will oscillate rapidly.

- An under-damped SHO is just like a free SHO except that the amplitude is damped by the time scale β. The oscillation is defined by an "envelope function" $Ae^{-\beta t}$.

- As β approaches ω_0, the oscillation becomes very very slow. $\omega_1 \to 0$ and T (period) $\to \infty$!

6.3.2 Over-damped SHO

For an over-damped SHO ($\omega_0 < \beta$), define

$$\gamma = \sqrt{\beta^2 - \omega_0^2}. \tag{6.33}$$

Then the solution becomes

$$x(t) = e^{-\beta t}\left[A_1 e^{\gamma t} + A_2 e^{-\gamma t}\right]. \tag{6.34}$$

Note that $0 < \gamma < \beta$, so that $\beta - \gamma > 0$. The two decay constants are, now, $\beta + \gamma$ (fast decay; the second term) and $\beta - \gamma$ (slow decay; the first term). There is no oscillation, but decay!

6.3.3 Critical damping

For a critically damped SHO ($\omega_0 = \beta$), we already wrote down the solution above:

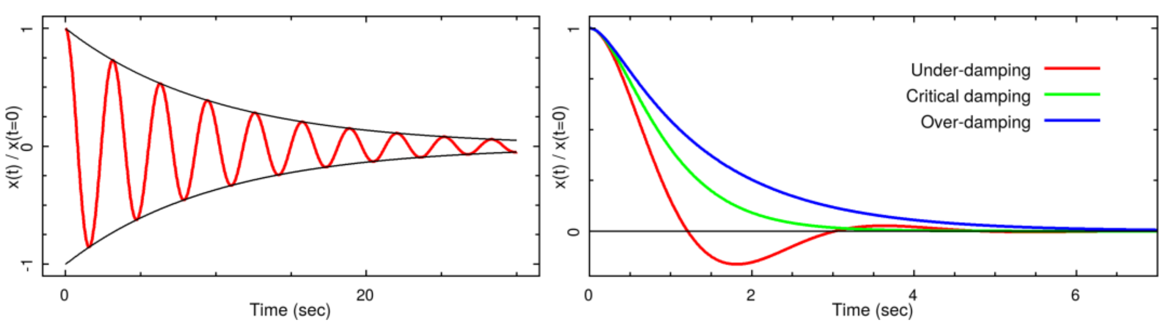
$$x(t) = e^{-\beta t}(A_1 + A_2\, t), \qquad \text{critical damping.} \tag{6.28}$$

This is the case when the decay behavior is governed by only one decay constant β. Clearly the over-damping would not be efficient in stopping a vibration than the critical damping, due to the slow decay term existing in the over-damped SHO solution. The critical damping is the best way to stop vibration.

6.3.4 Example plots

Let us make some plots considering simple cases when the oscillator is pulled to a certain initial position ($x(t=0) = x_0$), but the initial velocity is zero ($\dot{x}(t=0) = 0$). It is left for you to prove the following by matching these initial conditions.

1. (Under-damping case) $x(t) = Ae^{-\beta t}\cos(\omega_1 t + \phi_0)$, where $\phi_0 = -\tan^{-1}\frac{\beta}{\omega_1}$ and $A = \frac{x_0}{\cos\phi_0}$.
2. (Critical damping case) $x(t) = x_0 e^{-\omega_0 t}(1 + \omega_0 t)$.
3. (Over-damping case) $x(t) = e^{-\beta t}\left(A_1 e^{\gamma t} + A_2 e^{-\gamma t}\right)$ where $A_1 = \frac{1}{2}\left(1 + \frac{\beta}{\gamma}\right)x_0$ and $A_2 = \frac{1}{2}\left(1 - \frac{\beta}{\gamma}\right)x_0$.



In the above, two plots are shown. On the left is shown an under-damped SHO with $x_0 = 1$, $\dot{x}(t=0) = 0$, $\omega_0 = 2$ Hz, and $\beta = 0.1$ Hz. Since β is small, we get the oscillation (red line) is governed by $\omega_1 = \sqrt{\omega_0^2 - \beta^2} \approx \omega_0$. The black curves show $\pm e^{-\beta t}$, which acts as an **envelope function** that describes the overall decay of the amplitude.

The plot on the right shows the three different cases of damping with the same $\omega_0 = 2$ Hz and the same initial conditions ($x_0 = 1$ and no initial velocity). In the under-damped case, $\beta = 1$ Hz is chosen, while in the over-damped case $\beta = 3$ Hz is chosen. What this plot demonstrates is the fact that the damping is the best in the critical damping case.

6.3.5 Example—damped simple pendulum

Let us consider a simple pendulum that is placed in a fluid so that the resistance force, linear in the velocity, applies. The magnitude of the resistance force happens to be given by $2m\sqrt{\frac{g}{R}}v$, where v is the speed and R is the distance between the mass (m) and the pivot point. The pendulum has a small positive value, $\alpha > 0$, for the initial value of the angle θ, and zero velocity. What is the subsequent motion for $\theta(t)$ and what is the phase space trajectory?

Let us use the rotational form of the equation of motion, $I\ddot{\theta} = -mgR\theta + F_{resistance}R$. The first term is the torque due to the gravitational force: $-mgR\sin\theta \approx -mgR\theta$ for small θ ($|\theta| \ll 1$). $I = mR^2$. The resistance force is given by (since $\sqrt{\frac{g}{R}} = \omega_0$ and $v = R|\dot{\theta}|$) $F_{resistance} = -2m\omega_0 R\dot{\theta}$. So, the equation of motion is given by

$$mR^2\ddot{\theta} = -mgR\theta - 2m\omega_0 R^2\dot{\theta}, \tag{6.35}$$

$$\ddot{\theta} + 2\omega_0\dot{\theta} + \omega_0^2\theta = 0. \tag{6.36}$$

This is a critically damped oscillator problem ($\beta = \omega_0$).

$$\theta(t) = (A + Bt)\exp(-\beta t).$$

The initial condition is such that $A = \alpha$.

$$\dot{\theta}(t) = (B - \beta A - \beta Bt)\exp(-\beta t).$$

Since $\dot{\theta}(0) = 0$, $B = \beta A = \alpha\beta$. So, the solutions are given by

$$\theta(t) = \alpha(1 + \beta t)\exp(-\beta t), \qquad \beta = \omega_0 \tag{6.37}$$

$$\dot{\theta}(t) = -\alpha\beta^2 t\exp(-\beta t). \tag{6.38}$$

By plotting these two functions up, one can get both $\theta(t)$, and the phase space trajectory $(\theta(t), \dot{\theta}(t))$.

We can do a bit more work to understand the phase trajectory. The EOM can be re-written as (using the work energy theorem $\tau d\theta = -d(\frac{1}{2}I\dot{\omega}^2)$, where τ is the torque, or the differential calculation trick, $\ddot{\theta} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt}$):

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} + 2\omega_0\dot{\theta} + \omega_0^2\theta = 0,$$

from which we get

$$\frac{d\dot\theta}{d\theta} = -2\omega_0 - \omega_0^2 \frac{\theta}{\dot\theta}.$$

From the above solution, this means

$$\frac{d\dot\theta}{d\theta} = -2\omega_0 + \omega_0^2 \frac{1+\omega_0 t}{\omega_0^2 t} = -\omega_0 + \frac{1}{t} \tag{6.39}$$

where $\beta = \omega_0$ is used on substituting the above solution. One can see that the slope of the path in the phase space is initially ∞ as the right hand side diverges at $t = 0$. This is because initially the effect of resistance is near zero, and the path is an ellipse (or a circle if $\dot\theta$ is rescaled properly). As $t \to \infty$, the slope approaches $-\omega_0$. Also, it should be noted that $\theta > 0$ never changes sign, given the current initial condition. And, neither does the velocity $\dot\theta < 0$. And so, in the phase, the trajectory starts vertically down, before bending to the left. At time $t = 1/\omega_0$, the trajectory hits the minimum point, and after it bounces up and left (since $d\dot\theta/d\theta < 0$ for $t > 1/\omega_0$ and $|\dot\theta|$ has a maximum at $t = 1/\omega_0$ (Eq. 6.38)). Soon, it approaches the origin with a negative slope $-\omega_0$. The trajectory never leaves the fourth quadrant.

Here is a computer generated plot from the above solutions for $\theta(t)$ and $\dot\theta(t)$, with $\alpha = 1$ rad, $\beta = 2$ Hz. It has all the features that we just discussed.