Notes for Lecture 5

Conservation principles and 1D motions

So far, we have discussed the general structure of Newton's laws and some example solutions. We must not forget to discuss conservation principles, which cannot be over-emphasized. Here, we discuss these principles from Newton's law points of view. This is an elementary and old view. The discussion here is something of a reminder of these principles that you learned this way in an introductory course. Soon in this course, we will learn a much more powerful and general way to discuss conservation principles. Then, we will discuss an important class of problems in one dimension.

5.1 Conservation theorems

It is not possible to overemphasize the importance of the principle of conservation. In classical mechanics, these principles can be thought as consequences of Newton's laws. So, we call them conservation theorems. However, we will revisit and enrich these principles later. When we do, we will see that these principles go beyond Newton's laws.

5.1.1 Momentum conservation

$\dot{\vec{p}} = \vec{F}$ and so $\vec{p}$ is conserved when the net force is zero. It can happen that only certain components of the net force is zero. Then, only the corresponding momentum components are conserved. An example is the one that we worked out in the previous lecture: the motion of a charge in a magnetic field. While the total momentum is certainly not conserved, the z component (parallel to $\vec{B}$) of the momentum is conserved since the z component of the force is zero.

5.1.2 Angular momentum conservation

$$\vec{L} \overset{def}{=} \vec{r} \times \vec{p}$$

Of the two terms that result on taking the time-derivative of $\vec{L}$, the term $\dot{\vec{r}} \times \vec{p}$ vanishes since $\dot{\vec{r}} = \vec{v}$ is parallel to $\vec{p} = m\vec{v}$ (recall that $\vec{A} \times \vec{A} = 0$). As a result,

$$\dot{\vec{L}} = \vec{r} \times \vec{F} \overset{def}{=} \vec{N} \text{ (torque)}$$

So, $\vec{L}$ is conserved when the net torque is zero. It can happen that only certain components of the net torque is zero. Then, only the corresponding angular momentum components are conserved.

5.1.3 Mechanical energy and its conservation

The total mechanical energy is conserved if all forces are conservative. Special attention will be paid to the case when the total mechanical energy can be written as

$$E = T + U$$

We now define what T, U, and conservative force mean. In this course, we may drop "mechanical" in "mechanical energy."

- ***T* and work-energy theorem** Work (dW) done on an object by an applied force $\vec{F}$ is defined as

  $$dW \overset{def}{=} \vec{F} \cdot d\vec{r}$$

  This definition is valid for any force, not just the net force. Now, consider a motion of a particle moving from point 1 to point 2, and the total work done on it, W_{12}, during this movement. By total work, we mean work done by the net force, not by any individual force alone. For the net force, we can use Newton's equation, $\vec{F} = m\vec{a}$, and so the total work done on the particle is $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 m\frac{d\vec{v}}{dt} \cdot d\vec{r} = \int_1^2 m d\vec{v} \cdot \vec{v} = \int_1^2 \frac{m}{2} d(v^2) = \int_1^2 d(\frac{1}{2}mv^2) = T_2 - T_1$, where

  $$T \overset{def}{=} \frac{1}{2}mv^2$$

is the **kinetic energy**, and $T_1 = \frac{1}{2}mv_1^2$ and $T_2 = \frac{1}{2}mv_2^2$. To summarize,

$$W_{12}(\text{work done by net force}) = T_2 - T_1$$

This is the **work energy theorem**, which is *always valid*, not just when the net force is conservative. So, it applies when air resistance or friction is involved as well. Actually, a better way to look at it is that **the work energy theorem is just the way the notion of "kinetic energy" is defined, and why it has to be defined as** $mv^2/2$.

- ***U* and conservative force** $\vec{F}$ is a conservative force if $\oint_C \vec{F}\cdot d\vec{r} = 0$ for any closed path C. For instance, the Lorentz force is a conservative force, as is the gravitational force. For Lorentz force, $\vec{F}\cdot d\vec{r} = 0$. For the gravitational force, the positive work done by it and the negative work done by it cancel exactly in a closed path. So, a conservative force is in general a force that "gives back." However, air resistance or friction force is not a conservative force, as such a force does a negative work for any finite path, including a closed path. Such a force "only takes away" without giving back. An important type of force is a conservative force that depends on position only. In this case, the following three conditions are equivalent to one another.

$$\begin{aligned} \oint_C \vec{F}\cdot d\vec{r} &= 0 && \text{for any closed path } C \\ \vec{\nabla}\times\vec{F} &= 0 && \\ \vec{F} &= -\vec{\nabla}U(\vec{r}) && \text{where } U(\vec{r}) = -\int^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,' \end{aligned}$$

where U is the so-called **potential energy** function.[1] The integral notation $\int^{\vec{r}}$ used to define U means an indefinite line integral,[2] for which *any* path can be chosen as long as it ends at point $\vec{r}$. Note that if $U = U(\vec{r},t)$, i.e. an explicit function of time, then the force will not be conservative. This will happen, e.g., if the source of the gravitational force is in motion.[3]

[1] To prove the equivalence of the first two, use Stoke's theorem $\oint_C \vec{F}\cdot d\vec{r} = \int d\vec{s}\cdot\vec{\nabla}\times\vec{F}$. (To go from the first to the second, consider an arbitrarily small, but finite, closed loop integral.) To prove the equivalence of the first and the third, note that the vanishing loop integral means that $-\int_{\vec{r}_1,P_1}^{\vec{r}} \vec{F}\cdot d\vec{r} = -\int_{\vec{r}_1,P}^{\vec{r}} \vec{F}\cdot d\vec{r}$. Here, P_1 is taken as a *particular* path from $\vec{r}_1$ to $\vec{r}$, randomly chosen, while P is any path with the same end points. Since the RHS is independent of what the path P is as long as its end points are $\vec{r}_1$ and $\vec{r}$, we see that both sides are only dependent on the end points. Either side can be taken as $U(\vec{r})$.

[2] Recall from calculus that the definition of a line integral is an integral on a path (including a line, a curve, joined line segments, joined curve segments, etc.), *not* necessarily on a line.

[3] In this case, the force might still satisfy, at a *fixed* time, $\vec{\nabla}\times\vec{F} = 0$! For example, consider $F(\vec{r}) = -\hat{r}\alpha e^{-\gamma t}/r^2$, a gravitational force due to a hypothetical "dying star." This is not a conservative force, even if $\vec{\nabla}\times\vec{F} = 0$ at any *fixed* time t, since in physical motions t and $\vec{r}$ are dependent on each other, and so $\vec{\nabla}\times\vec{F} \neq 0$ if that dependence is taken into account.

- **Energy is only relative.** The definition of the potential energy is ambiguous up to a constant, because the line integral, $U(\vec{r}) = -\int^{\vec{r}} \vec{F} \cdot d\vec{r}\,'$, has an arbitrary starting point. The kinetic energy is also not an absolute concept, since, due to the Galilean invariance, the speed can be measured differently depending on which inertial frame is used. It follows then that the absolute value of energy is not meaningful. What matters is the *change of energy*. The energy conservation means that once the energy is defined in an inertial frame with a fixed zero reference, then the energy is constant as a function of time.

5.1.4 Constant or integral of motion

The importance of conservation principles cannot be overemphasized. When a certain physical quantity is conserved, it means that there is a function f, or a set of such functions, of $\vec{x}$ and $\dot{\vec{x}}$, which satisfies

$$f(\vec{x}, \dot{\vec{x}}) = \text{constant}$$

Here, f can be the energy function $T + U$, each/a component of the momentum function $m\vec{v}$, or each/a component of the angular momentum function $m\vec{r} \times \vec{v}$. Or, in general, it can be some other function (do a web search on Laplace-Runge-Lenz vector, for example). Each function f represents a conserved quantity, which is called a **constant of motion**, if the RHS of the above equation is emphasized, or an **integral of motion**, if the LHS of the above equation is emphasized.

1. **Constant of motion** We know that the general solution for Newton's equation should involve $2M$ integration constants (M = degrees of freedom). They correspond to the initial condition of position and velocity. A constant of motion due to a conservation principle is not any additional constant, but simply a function of those initial condition constants. Namely, one can re-express some of the initial condition constants in terms of constants of motion associated with conservation principles.
   For instance, a uniform circular motion (centered at the origin) can be specified by giving the radius and the initial phase (the last lecture). Equivalently, it can be specified by giving the energy and the initial phase. Or, the angular momentum and the initial phase.

2. **Integral of motion** Each integral of motion $f(\vec{x}, \dot{\vec{x}})$, if found, means "one less integration to do" for solving Newton's equation. This is why it is called an integral of motion. To illustrate this point, consider the general form of

Newton's equation in 2D.

$$m\ddot{x} = F_x(x, y, \dot{x}, \dot{y}, t)$$
$$m\ddot{y} = F_y(x, y, \dot{x}, \dot{y}, t)$$

If there is a constant of motion, $f(x, y, \dot{x}, \dot{y}) = \text{const}$, we can solve for, say, $\dot{x} = g(x, y, \dot{y})$, where g is the result of inverting $f(\dots) = \text{const}$. Therefore, the above Newton's equation can be rewritten as

$$m\frac{d}{dt}g(x, y, \dot{y}) = F_x(x, y, \dot{x}, \dot{y}, t)$$
$$m\ddot{y} = F_y(x, y, \dot{x}, \dot{y}, t)$$

Notice that the second derivative of x has completely disappeared, making it much easier to find an analytical solution, since we now have a first order differential equation for x, rather than a second order one. This is the meaning of "one less integration to do."

If a sufficient number of integration constants are recognized first, a seemingly complicated problem can become easy to solve and understand. **In fact, the first question one should ask when given any physics problem is "what are the integrals of motion?"** because of this reason.

5.1.5 Example—a mouse on a fan

Let us assume that a ceiling fan is rotating at an initial angular speed ω_0. A mouse jumps on the fan at radius R and velocity v (just before landing), where the direction of v is same direction as the velocity of the fan at the point of the landing. What would be the value of the angular speed ω of the fan, after the mouse has landed on it?

"Clearly," the angular momentum is conserved in this problem. $L_0 = I\omega_0 = L = I\omega + mvR = \frac{v}{R}(I + mR^2)$. $\therefore \omega/\omega_0 = I/(I + mR^2)$. Questions. "Clearly" usually does not cut it. *Why* is the angular momentum conserved in this problem? Why is the energy not conserved in this problem? What will happen if the (horizontal) ceiling fan was (vertical) windmill blades instead? Will the angular momentum conserved? Is the energy conserved during the collision? After the collision?

5.2 Stable or unstable equilibrium

If the potential satisfies $\vec{\nabla}U(\vec{r}) = 0$ at a certain point in space, then it means that the force is zero. If a particle can be placed at such a point with zero velocity, then the particle will stay there forever. Such particle is said to be in **equilibrium**. An important physical question is whether such an equilibrium will be stable or not. If a small perturbation from the environment pushes the particle away from the equilibrium by an ever so tiny displacement, will the state be robust enough to come back to the equilibrium point (stable) or not (unstable)?

So, the fundamental feature for a stable equilibrium point is the existence of the **restoring force** when the system is displaced slightly out of the equilibrium. A restoring force is a force that accelerates the particle towards the equilibrium point, not away from it. An elementary example is a spring force. Note that a restoring force is required for a displacement in *any* possible degree of freedom for the equilibrium point to qualify as a stable one. It follows that one can classify equilibrium points conveniently using the potential energy.

- If the potential is minimum[4], then it is a stable equilibrium. This is because, in the immediate vicinity of a minimum potential, the force points towards the minimum point.[5]
- If the potential is maximum, then it is an unstable equilibrium. This is because in the immediate vicinity of a maximum potential, the force points away from the maximum point.
- If the potential is neither maximum nor minimum ("saddle point"), then its stability is direction dependent.

It follows that, in 1D, if the second derivative is positive, then it means a stable equilibrium. If negative, then an unstable equilibrium. If zero, then examine higher derivatives, or perhaps plot the potential, to figure out whether it is a minimum, a maximum, or a saddle point.

Let us recall the following simple points. In a linear 1D motion, the restoring force for a stable equilibrium point takes the form of $F = -dU/dx = -kx$ (assuming that the equilibrium point of x is zero). Then it follows from Newton's law: $m\ddot{x} = -kx$ that the quantity $\omega = \sqrt{k/m}$ gives the "angular frequency" of the resulting **simple harmonic motion**. Similarly, a motion that involves a pure rotation around one axis can be taken as a rotational 1D motion, since it involves one angular coordinate

[4] A sufficient condition for this is that all eigenvalues of the matrix $\frac{\partial^2 U}{\partial x_i \partial x_j}$ is positive, in the most general case when there are M x_i variables involved, where M is the total degrees of freedom. This can be proven easily since $\frac{\partial^2 U}{\partial x_i \partial x_j}$ is a real symmetric matrix.

[5] Recall from calculus that the direction of $\vec{\nabla} U(\vec{r})$ is the direction in which U increases the most, when a small displacement $\vec{r} \to \vec{r} + d\vec{r}$ is considered with fixed $|d\vec{r}|$. So, near the minimum point $\vec{\nabla} U$ points away from the minimum, and thus the force $\vec{F} = -\vec{\nabla} U$ points towards the minimum point.

θ. If a conservative force is in operation (as in a simple pendulum problem or a physical pendulum problem), then the potential energy can be written as $U(\theta)$, and the "restoring force" in this case is rather a restoring torque: $\tau = -dU/d\theta = -\kappa\theta$ (assuming again that the equilibrium point of θ is zero). Then, from the rotational form of Newton's law, $I\ddot{\theta} = -\kappa\theta$, and $\omega = \sqrt{\kappa/I}$, where I is the rotational inertia. In all of these, the "angular" in the term "angular frequency" does not mean any thing in terms of the real space motion. As we will see later, a simple harmonic motion is a projection of a circular motion in a complex plane, and this is where the term "angular" comes from.

5.3 Example—a double well potential

Let us consider a potential given by

$$U(x) \;=\; -W\frac{d^2(x^2+d^2)}{x^4+8d^4}. \tag{5.1}$$

What can one say about the nature of motions possible for this potential? What are the turning points of the motion, if any, if $E = -\frac{W}{8}$?

We can begin by re-writing the above function in a nicer way.

$$\begin{aligned}
Z(y) \stackrel{def}{=\!=} U(x)/W &= -\frac{y^2+1}{y^4+8} && y \stackrel{def}{=\!=} x/d \\
dZ/dy &= -\frac{2y}{y^4+8} + \frac{(y^2+1)(4y^3)}{(y^4+8)^2} \\
&= \frac{4y^5+4y^3-2y(y^4+8)}{(y^4+8)^2} \\
&= \frac{2y^5+4y^3-16y}{(y^4+8)^2} \\
&= 2y\frac{y^4+2y^2-8}{(y^4+8)^2}
\end{aligned}$$

The zeroes of dZ/dy are at $y = 0, \pm\sqrt{2}$. $Z(y)$ is an even function. It is very useful to examine various limiting cases.

$$\begin{aligned}
Z(y) &\approx -(1+y^2)/8, && y \approx 0. \\
Z(y=\pm\sqrt{2}) &= -\frac{1}{4}, \\
Z(y) &\approx -1/y^2, && |y| \to \infty.
\end{aligned}$$

So, it is clear that $Z(y)$ is an inverted parabola near $y = 0$. This means that $y = 0$ is the maximum point for $Z(y)$. Consider increasing y from 0. The next extremum value is at $y = \sqrt{2}$, where Z has zero slope. This must be a minimum since $Z(y = \sqrt{2}) < Z(y = 0)$ or $Z(y = \infty)$. So, the function hits the minimum at $y = \sqrt{2}$, before increasing to an asymptote 0 at $y = \infty$. The mirror image behavior occurs for the negative y part. So, this is a "double well" potential.

1. The $x = 0$ point ($y = 0$) is an unstable equilibrium point with $E = -\frac{W}{8}$.
2. Two stable equilibrium points are found at $x = \pm\sqrt{2}d$.

The turning points at energy $-W/8$ is obtained by $Z(y) = -1/8$, which means

$$y^4 + 8 - 8y^2 - 8 = 0$$

So, we get $x = \pm 2\sqrt{2}d, 0$. Now, what would be the period of the motion at this energy $E = -W/8$?

5.4 One dimensional motion

As we will see in this course, many problems become effectively one dimensional in certain ways, so it is worthwhile to summarize this simple case. We consider a 1D motion in the presence of a potential energy $U(x)$. So, this is a 1D motion with a conservative force. Because the energy is then an integral of motion, the 2nd order equation becomes the 1st order equation, for which the formal solution can be written down with ease.

For a given potential energy, the physical motion is possible only when

$$E \geq U(x)$$

(this is true in any dimensions, actually; it is because $T = E - U \geq 0$). For a given energy value E and the initial position x_0, the motion can be **bound** or **unbound** depending on how many **turning points** are encountered as the x value is changed continuously from x_0. A bound motion has two turning points. An unbound motion has one or no turning point. However, a turning point occurring at an unstable equilibrium point should be treated as an exception. The diagram below illustrates the turning points (centers of large circles). Note that a bound motion or an unbound motion can occur at the same energy (e.g., E_3).

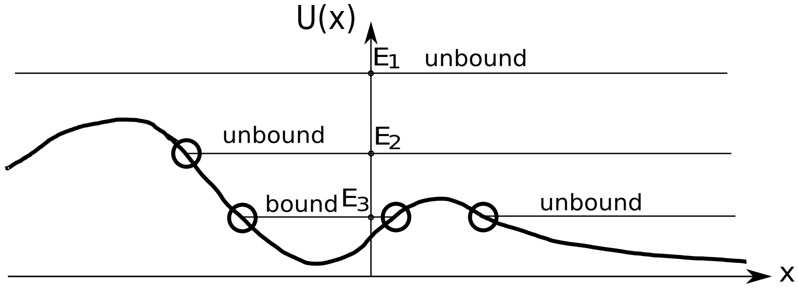
Solution for 1D problems with $U(x)$

When a problem in 1D is described by a potential energy $U(x)$, its general solution can be written down as an integral.

$$
\begin{aligned}
E &= \frac{1}{2}mv^2 + U(x) = \text{constant} \\
v &= \pm\sqrt{\frac{2}{m}(E - U(x))} = \frac{dx}{dt} \\
\therefore\ t &= \pm\int_{x_0}^{x} dx' \sqrt{\frac{m}{2(E - U(x'))}}
\end{aligned}
$$

The last equation is the general solution of a 1D problem with the potential energy $U(x)$. The two integration constant symbols are E and x_0. The fact that the solution occurs in pair with a different overall sign of t ($\pm$) means that, if $x(t)$ is a solution, then its "time-reversed" solution $x(-t)$ is also a solution.[a] For a periodic motion, it is redundant to keep the time-reversed solution, because reversing time is equivalent to shifting time (please convince yourself of this).

[a] Therefore, this problem has the "time-reversal symmetry." It comes from the fact that the EOM, $md^2x/dt^2 = -dU(x)/dx$, is invariant, when the sign of t is inverted.



Any bound motion in 1D, where all forces are conservative, is a periodic motion. The period is given by

$$
T = 2\int_{x_{t,1}}^{x_{t,2}} dx' \sqrt{\frac{m}{2(E - U(x'))}}
$$

where $x_{t,1}$ and $x_{t,2}$ are the two turning points[6].

[6] If two turning points include unstable equilibrium point(s), then this must be taken as an exception, since the period goes to infinity.