Notes for Lecture 4

Simple Lorentz force

In this Lecture, we shall consider a kind of simple, but important, problem—the problem of a charge in a uniform $\vec{B}$ field.

4.1 Motion of a charge in a uniform magnetic field

Let us assume that there is a magnetic field pointing along the z direction. $\vec{B} = B_0\hat{z}$, where $\hat{z}$ is the unit vector along the z direction. The Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$, where q is the electric charge of the particle. Here we consider a simple example, where $\vec{B}$ is constant.

Any vector product can be calculated, if the following basic rules are noted, along with other usual properties of multiplication (associative rules, distributive rules, ...).

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \qquad \textit{anti}\text{-commutative!} \tag{4.1}$$

$$\hat{x} \times \hat{y} = \hat{z} \tag{4.2}$$

$$\hat{y} \times \hat{z} = \hat{x} \tag{4.3}$$

$$\hat{z} \times \hat{x} = \hat{y} \tag{4.4}$$

All of the following properties can be derived from the above basic rules. Knowing these by heart will server you well.

1. $\vec{A} \times \vec{A} = 0.$

2. **Right screw/hand rule** applies. Rotate $\vec{A}$ towards $\vec{B}$ (involving the shorted angular displacement possible). How would a right-handed screw move along its axis on such a rotation? That is the direction of $\vec{A} \times \vec{B}$.

3. $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$: $(\vec{A} \times \vec{B}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot \vec{B} = 0$.

4. $|\vec{A} \times \vec{B}|$ is the **area of the parallelogram** spanned by $\vec{A}$ and $\vec{B}$, i.e. twice the area of the triangle formed by $\vec{A}$ and $\vec{B}$. It is $AB \sin\theta$.

5. $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \hat{i} A_j B_k$, where ε_{ijk} is the Levi-Civita symbol ($\varepsilon_{123} = 1$ and it changes sign whenever two indices are swapped, i.e. permuted), and $\hat{1} \overset{def}{=} \hat{x}$, $\hat{2} \overset{def}{=} \hat{y}$, $\hat{3} \overset{def}{=} \hat{z}$. Here, the middle term means the determinant of the 3×3 matrix. Strictly speaking this matrix does not make sense as its elements must be numbers, not vectors. However, as *mnemonics*, this matrix serves us well.

6. In other words, $(\vec{A} \times \vec{B})_3 = A_1 B_2 - A_2 B_1$ and, by **cyclic permutations of indices**, $(\vec{A} \times \vec{B})_2 = A_3 B_1 - A_1 B_3$ and $(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$.

7. $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$. The magnitude $|(\vec{A} \times \vec{B}) \cdot \vec{C}|$ is the volume of the parallelepiped formed by these three vectors.

8. Finally, an essential identity! $\sum_{k=1}^{3} \varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$.

The Lorentz force is then $q\vec{v} \times B_0 \hat{z} = q(v_y B_0 \hat{x} - v_x B_0 \hat{y})$. Notice that there is no force along the z direction, since $\hat{z} \times \hat{z} = 0$. So, $v_z = v_{z,0}$.

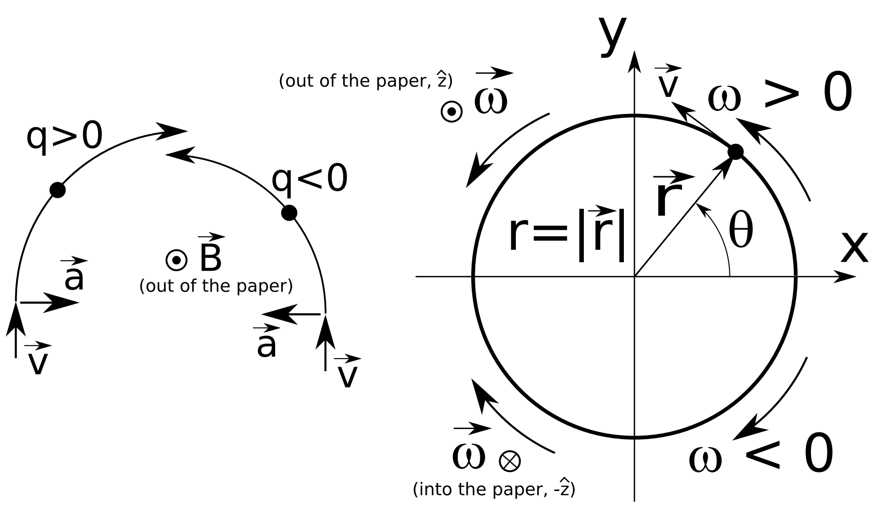
Let us use the inertial frame, which is moving at $v_{z,0}\hat{z}$. Then, there is no z motion, and we can simply deal with x and y motions only.

$$ma_x = qv_y B_0 \tag{4.5}$$
$$ma_y = -qv_x B_0 \tag{4.6}$$

$$a_x = -\omega_c v_y \qquad \omega_c \stackrel{def}{=\!=} -qB_0/m \tag{4.7}$$
$$a_y = \omega_c v_x \tag{4.8}$$

$$\vec{a} = \omega_c(-v_y \hat{x} + v_x \hat{y}) \tag{4.9}$$
$$\vec{a}\cdot\vec{v} = 0 \qquad \omega_c(-v_y v_x + v_x v_y) = 0 \tag{4.10}$$

Here, $\omega_c = -qB_0/m$ has the dimension of inverse time, and it is call the "**cyclotron frequency**." The motion is a **uniform circular motion**, with the angular frequency given by ω_c. How do we know this? (1) The acceleration is always perpendicular to the velocity. So, it is a centripetal acceleration. There is no tangential acceleration[1], so v will be constant. So, a uniform circular motion. (2) The magnitude of the acceleration is, $a = \sqrt{\omega_c^2 v_x^2 + \omega_c^2 v_y^2} = |\omega_c| v$. This is precisely the uniform circular motion relation that you learned in an elementary course, with the angular frequency $|\omega_c| = 2\pi/T$ where T is the period[2].



These arguments are quite satisfactory, but let us do some more work here, now that we know precisely what it means to solve a Newton's equation. Notice that we have a 2D problem of one particle, so the number of integration constants (i.e., those constants for specifying the initial condition; see below for more information) is four.

[1] $\vec{a}\cdot\vec{v} = 0$ means this. If you like, then the following math will help. $\frac{d}{dt}v^2 = \frac{d}{dt}\vec{v}\cdot\vec{v} = 2\vec{a}\cdot\vec{v} = 0$

[2] Unfortunately, T is a versatile symbol, used for multiple purposes. Sometimes it is "time" (dimension). Sometimes it is period. Sometimes it is kinetic energy. Or, temperature. Usually the context makes the meaning unambiguous.

How many numbers do we need to characterize a uniform circular motion? 2 for the position of the center. 1 for the radius r. 1 for the speed v. 1 for the initial phase (θ_0),[3] the initial angular position. It seems that we have 5—which is one too many. But, remember that $v = 2\pi r/T$, and so r and v are physically constrained by ω_c, i.e. by B_0 and q/m. So we have 4 numbers. Good. As the origin of the coordinate system is arbitrary, let us choose the most convenient coordinate system so that the center of the circle is the origin. Then, we expect to have two integration constants in the solution.

Now, we are ready to write down the solution. By using the above coordinate system, and identifying $\theta = \omega_c t + \theta_0$, we get

$$x = r\cos(\omega_c t + \theta_0) \tag{4.11}$$

$$y = r\sin(\omega_c t + \theta_0) \tag{4.12}$$

Here, r, θ_0 are two integration constants that need to be fixed by the initial condition. Namely, those are the parameters that appear in the solution in addition to symbols defined in the equation of motion (Eqs. 4.5 and 4.6, together, or Eq. 4.9 alone). Upon taking the derivative, $\dot{x} = -\omega_c y$, and $\dot{y} = \omega_c x$. Repeating, $\ddot{x} = -\omega_c \dot{y}$, and $\ddot{y} = \omega_c \dot{x}$. This agrees with the equations that we wrote above ($a_x = -\omega_c v_y$ and $a_y = \omega_c v_x$), i.e., Eq. 4.9. This ends the proof that the above solution is the general solution.

If we include the z motion, then the motion is that or a cylindrical spiral motion, a circular motion in the $x - y$ plane plus a uniform translation along the z axis.

Notice that we defined ω_c so that it is negative for the positive charge and positive for the negative charge. This is the result of our setting up the coordinate system as shown above.

Note that the angular velocity vector $\vec{\omega}$ is in general defined through a right-hand rule. We will study more details later, but, at this point, it should suffice to study the direction of $\vec{\omega}$ in the above diagram.

As you can see, the cyclotron motion can be useful to figure out what the sign of the particle's electric charge is, or what the kinetic energy of the particle is after a collision, if q/m is known.

Lastly, note that the time reversal symmetry is broken for this problem. What this means is that when the motion is played backwards (without changing the direction of $\vec{B}$: often the direction of $\vec{B}$ is thermodynamically determined and it is a given, non-reversible, or it involves a dissipative process and it is non-reversible), then an impossible motion results (e.g. a positive charge doing a clock-wise rotation will results in a positive charge doing a counter-clock-wise rotation upon the timer reversal).

[3] The "phase" is defined as θ. And so, θ_0 is the *initial* phase.

However, different from dissipative problems (friction, air resistance), which also break the time reversal symmetry, the mechanical energy is conserved here. The Lorentz force is perpendicular to the velocity, and so it actually **does not do any work**.

4.2 Some more thoughts—UCM and SHM

In the above, we simply wrote down solutions, Eqs. 4.11 and 4.12, showed that they are indeed the solutions, and they have the correct number of integration constants. This may seem like a devious thing to do, but given the framework that was explained in the previous lecture, this is a quite a respectable way to solve the problem.

Even so, it may seem a bit opaque. Perhaps using another point of view of this problem may help.

In the above, we saw that $a_x = -\omega_c v_y$ and $a_y = \omega_c v_x$. Then, by taking the time derivative to the second equation, we get $\dot{a}_y = \omega_c a_x$, where $\dot{v}_x = a_x$ is used. Using $a_x = -\omega_c v_y$, then, we get[4]

$$\ddot{v}_y = -\omega_c^2 v_y. \tag{4.13}$$

This is a quite familiar equation of motion to you, I hope! We have already seen it in Lecture 3, near the end, and it is the simple harmonic motion equation of motion. Normally, it is the equation of motion for x, but this one is written in terms of v_x. In any case, the general solution is given by $v_y = A\cos(\omega_c t + \theta_0)$. Integrating, we get

$$y(t) = r\sin(\omega_c t + \theta_0) + y_0. \tag{4.14}$$

Here, we defined[5] $r \equiv A/\omega_c$. Taking y_0 as zero, we see that this one is identical with Eq. 4.12. Also, since $v_x = a_y/\omega_c$, we get $v_x = -r\omega_c \sin(\omega_c t + \theta_0)$, and this gives

$$x(t) = r\cos(\omega_c t + \theta_0) + x_0. \tag{4.15}$$

Again, by taking $x_0 = 0$, as we did in the previous section, we see that we get exactly the same result as Eq. 4.11.

Now, let us pause and think what our solutions mean. In elementary physics, you might have learned that a simple harmonic motion (SHM) is a projection of a uniform

[4]It is of course possible to obtain SHM equations for $y' = y - y_0$ or $x' = x - x_0$, as well, by first integration $a_x = -\omega_c v_y$ or $a_y = \omega_c v_x$. This is left for your exercise.

[5]Note that in this lecture note ω_c is a signed number. This means that $r = A/\omega_c$ can be negative if $A > 0$ and $\omega_c < 0$. However, we can always shift θ_0 by π to change the sign of A: this way, we can always guarantee that $r > 0$.

circular motion (UCM). This problem gives a concrete example for this! The circular motion is caused by a constant magnetic field, and we can look at the equations of motion in two different ways—that of the UCM ($\vec{a} \cdot \vec{v} = 0$) and that of the SHM.