Notes for Lecture 2

Newton's laws

2.1 Newton's laws

Velocity vector

$$\vec{v} \stackrel{def}{=\!=} \frac{d\vec{r}}{dt} \stackrel{def}{=\!=} \dot{\vec{r}}$$

Here, we are introduced to the concept of the derivative of a function ($\vec{r}(t)$) with respect to its variable (time t). The first notation is due to Leibniz, who discovered calculus about the same time as Newton. In the second form, the dot on function is Newton's short-hand for the time derivative. Newton's notation is compact, but the Leibniz notation is arguably richer. The chain rule, such as

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt}$$

is very natural in Leibniz notation, as dg just "cancels out." Also, one often writes

$$\frac{df}{dt} = \frac{d}{dt}f = \left(\frac{d}{dt}\right)f$$

where $\left(\frac{d}{dt}\right)$ is now easily recognized as an **operator** on a function. Moreover, it is a **linear operator** since

$$\frac{d}{dt}(af(t) + bg(t)) = a\frac{d}{dt}f + b\frac{d}{dt}g$$

for any constants a, b and any functions f, g. Compare this definition with the fundamental definition of a linear coordinate transformation of the previous lecture (Equation 1.3 of page 6), and note the similarity.

Acceleration vector

$$\vec{a} \stackrel{def}{=} \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}} = \left(\frac{d}{dt}\right)^2 \vec{r}$$

Note that here, the **second-derivative operator** $\left(\frac{d}{dt}\right)^2$ is also a **linear operator**.

Newton's first law A particle with a constant velocity will remain at that velocity if not acted upon by a force. (Law of inertia)

Newton's second law A particle acted upon by a force, $\vec{F}$, will change its motion, according to

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{2.1}$$

where

$$\vec{p} = m\vec{v}$$

is the **momentum** of the particle. If the mass is constant, then

$$\vec{F} = m\vec{a}. \tag{2.2}$$

Newton's third law If particle 1 exerts a force $\vec{F}_{21}$ on particle 2, then particle 2 exerts an equal and opposite force $\vec{F}_{12} = -\vec{F}_{21}$ on particle 1. ("Action-reaction")

What do Newton's laws really mean? We can take it to mean that the first two laws are the definition of "force," while the "mass" is taken as a known quantity (i.e. measured with a scale—the principle equivalence would guarantee that the **gravitational mass**—the mass measured as the weight divided by $\vec{g}$—is the same as the **inertial mass** – the mass that enters in Newton's 2nd law). Then, really the third law is the only physical law. A different view is also possible (see discussion in various books).

Notice that the third law is equivalent to the **conservation of momentum** in the following sense. Consider two particles as an isolated system. Then, the total force acting on this system is $\vec{F}_t = \vec{F}_{21} + \vec{F}_{12}$. The total momentum, $\vec{P} = \vec{p}_1 + \vec{p}_2$, will then change according to the 2nd law: $\dot{\vec{P}} = \vec{F}$. So, one can see that $\vec{P}$ = constant (conserved) if and only if $\vec{F} = 0$, i.e. $\vec{F}_{21} = -\vec{F}_{12}$.

Newton's 3rd law fails?!. . . (optional reading)

You may hear or read that Newton's third law fails (or "appears to fail") in certain situations (e.g. when two moving charges interact in a certain geometry—do your Google search on this, if you are curious). It would seem that when such an example is discussed, a presumably classical electro-magnetic problem is invariably imagined. But some thoughts given to such a "classical" problem would convince any reasonable person that it would be quite impossible to consider such an experiment without worrying about other many body effects, such as screening and polarization, that must happen for classical systems. In order to help any of you, who might be interested in this type of discussion (which, by the way, I have to warn you a little against; at any given point of time, there seem to be many far-fetched or "revolutionary" claims in science, and many of them do not really pan out. This does not mean science is bad; it is quite the opposite), here I will state some guiding principles to keep in mind.

1. The momentum conservation is a much more general principle than Newton's 3rd law.

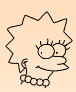
2. If the momentum is conserved and if any classical system can be divided into two classical particles exchanging forces, then Newton's 3rd law *must* follow, as we just showed.

3. If the momentum is conserved and if 2 fails, then it follows that the assumption of the "two separate classical particles" is invalid. Instead, quantum mechanics should be used. Newton's laws are meaningless to discuss.

Please read my note at the link "Nature-Laws" on my 2010 course web site for some more related discussion. Also, read Feynman, vol. II, chapter 28.

Inertial reference frame A reference frame is another name for the coordinate system. A reference frame in which Newton's laws are valid is called an inertial reference frame.

I would invite you to think about what this definition really means. You should think of experiments you did in your laboratory course, and how you could verify Newton's laws. It is important to realize that there is no such thing as an absolute inertial reference frame.



Galilean invariance

Suppose there is an inertial reference frame, and another reference frame that is in uniform motion relative to that. Then, the second reference frame is also an inertial reference frame. This is the **relativity principle** at the classical mechanics level. Roughly put, the relativity principle means that physics, or physical laws, are independent of reference frames. Oohoo!

Since the origin of the second reference frame is moving at a constant velocity relative to the first, $\vec{r}\,' = \vec{r} + \vec{V}t + \vec{R}_0$ where $\vec{R}_0$ is a constant position vector and $\vec{V}$ is a constant velocity vector. Note that $\ddot{\vec{r}}\,' = \ddot{\vec{r}}$. Thus, Newton's 2nd law in the first frame, $\vec{F} = m\ddot{\vec{r}}$ means $\vec{F} = m\ddot{\vec{r}}\,'$.

Free body diagram Please study this basic subject carefully, if you are not sure about this, using either your old textbook or references. I am not covering this basic topic, in the interest of time. We will be practicing free body diagrams later though.

Classical Particle?

Did you also notice that there is a circular structure to Newton's laws? (i) Consider a particle with mass m_1. No acceleration? That means no force. Acceleration? That means force. If it sees another particle with mass m_2, then it exchanges 3rd law pair forces. (ii) Now, group these two masses together and call it a "particle" with mass M_1 ($= m_1 + m_2$). Consider it as a point. No acceleration of M_1? It means no *external* force! This law of inertia is now a *derived* property that depends on the last step of (i). Acceleration? Must be an external force. If M_1 sees another particle with mass M_2, then they will exchange 3rd law pair forces. This is again a derived property, now. (iii) Now, group these two masses, M_1 and M_2 as a "particle." Consider it as a point. The law of inertia for this new particle is again guaranteed by the last step of (ii). And on and on.

While this circular structure is a sure sign for the self-consistency of the theory, it also means a certain lack of starting point. Indeed, the initial "particle" in the above reasoning is also a macroscopic object consisting of many fundamental particles of Nature. If this initial particle is divided up too finely, Newton's laws break down. How small can the classical particle be? At this point, it is crucial to keep in mind that all physics theories are based on experiments. The answer is "it depends." It depends on materials and conditions such as the temperature, and we need to rely on experiments on a case by case basis to know for sure.

2.2 Solving Newtonian equation of motion

For a one body problem with a constant mass, we need to solve

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}.$$

Let us consider a 1D problem first.

$$F = m\ddot{x}.$$

This is a so-called **second order differential equation** for $x(t)$, since it involves a second derivative but no higher derivative.

$F = F(x, \dot{x}, t)$, as far as we know. I.e., F is an explicit function of position, velocity and time, but no higher order derivatives of x.

This makes the above equation completely solvable. How? Like the computer does. There is one condition, though. We need to know x and $\dot{x}$ at the initial time (**initial condition**). Say, $x_0 = x(t = 0), v_0 = \dot{x}(t = 0)$. Then, $x(t)$ at any time is known! (**Newtonian determinism**)

Here is how it goes.

$$
\begin{aligned}
x_0, v_0 &\xrightarrow{\text{determine}} F_0 \stackrel{def}{=} F(x_0, v_0, t = 0) \\
F_0 &\xrightarrow{\text{gives}} a_0 = F_0/m \\
x_0, v_0 &\xrightarrow{\text{determine}} x_\epsilon \stackrel{def}{=} x(t = \epsilon) = x_0 + v_0\epsilon \\
v_0, a_0 &\xrightarrow{\text{determine}} v_\epsilon \stackrel{def}{=} v(t = \epsilon) = v_0 + a_0\epsilon \\
x_\epsilon, v_\epsilon &\xrightarrow{\text{gives}} F_\epsilon \stackrel{def}{=} F(x_\epsilon, v_\epsilon, t = \epsilon) \\
x_\epsilon, v_\epsilon, F_\epsilon &\xrightarrow{\text{repeat}} x_{2\epsilon}, v_{2\epsilon}, F_{2\epsilon} \\
x_{2\epsilon}, v_{2\epsilon}, F_{2\epsilon} &\xrightarrow{\text{repeat}} x_{3\epsilon}, v_{3\epsilon}, F_{3\epsilon} \\
&\xrightarrow{\text{and repeat}} \\
&\ldots \\
&\xrightarrow{\text{we can get}} x(t), v(t) \text{ for } \textit{any} \text{ time } t!
\end{aligned}
$$

This basically summarizes the concept of calculus, of course. Divide the finite time interval into many a pieces of very very short of intervals, where the linear approximation is practically precise, and put all pieces together. ϵ is the infinitesimal, which means a very small finite number, for which the Taylor series approximations $x(t = \epsilon) \approx x_0 + v_0\epsilon$ and $v(t = \epsilon) \approx v_0 + a_0\epsilon$ become *identities*, as used above, within the error bars of the finest measurement that we can make.

Why are the two constants, x_0 and v_0, necessary? Because we are doing a double integration, due to the differential equation being a second order equation. So, two constants are necessary for the most general solution. We say that there are **two integration constants** in solving a motion of a particle in one dimension. These constants may be fixed by any two independent physical conditions given in an actual problem.

The above procedure is readily generalized to higher dimensions as well. All we need to do is to put a vector sign on symbols, x, v, and F.

$$\vec{F}(\vec{x}, \vec{v}, t) = m\vec{a}.$$

The key steps are

$$\begin{aligned} \vec{v}(t = \epsilon) &= \vec{x}_0 + \vec{v}_0 \epsilon \\ \vec{x}(t = \epsilon) &= \vec{v}_0 + \vec{a}_0 \epsilon \end{aligned}$$

In this case, there will be **precisely** $2D$ **integration constants**, $\vec{x}_0$ and $\vec{v}_0$.

Seeing that the Newton's second law equation of motion can be solvable for any given force is excellent. This type of solution, if obtained with the help of a computer, is called a **numerical solution**. For some important classes of problems, we can find the solution with paper, pencil and some math tricks. Such a solution is called an **analytical solution**, which is what we will spend most of our time on in this course.

For a given initial condition and a given force, the above procedure guarantees a unique solution. This fact leads to the following observation.

What it means to solve Newton's equation

Suppose we are given a Newton's equation with the total degrees of freedom $= M$. If we somehow manage to find an analytical solution which include $2M$ integration constant symbols, then that is *the* general solution! Conversely, the general solution for such an equation should have exactly $2M$ integration constant symbols. It does not matter how we find the solution (even if it was a guesswork ... mmm ... I mean an *educated* guesswork!). The $2M$ integration constant symbols would be replaced with numerical values, if the initial condition or equivalent physical condition were specified.

Linear or non-linear

As we saw above, $\vec{a} = \ddot{\vec{x}}$ is a linear operator $(d/dt)^2$ acting on the function $\vec{x}(t)$. In other words, $\vec{a}$ is linear in $\vec{x}$. So, in $\vec{F} = m\vec{a}$, the right hand side is linear in $\vec{x}$. If the force is also linear in $\vec{x}$, then we say that we have a linear equation of motion. If not, then we have a non-linear equation of motion. We will see later that non-linear Newton problem can be quite interesting! At this point, though, let us just make sure that we understand that $\vec{F} = m\vec{a}$ is always solvable, linear or non-linear.

Solvability and integrability

Newton's equation of motion (2nd law) is *always* solvable, in principle,[a] by a computer.
Did you notice that the expression "**integrable**" is used for Newton's equation? That means "analytically integrable" since Newton's equation is always integrable numerically. When it is analytically integrable, it means that a closed form solution can be found "by hand."

[a] Well, there is a serious question of how big a computer you need and how much time you need, when many particles are involved. That is for discussion at another time.

2.3 An example—air resistance

Let us consider a particle moving along one direction, and let us assume that the only force that is acting on it is air resistance: $F = -kmv$, $k > 0$ (which is the general behavior of the air resistance at low speeds). What are $x(t), v(t)$?

Equation of motion:

$$m\dot{v} = -kmv$$

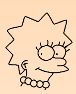
We have a rather happy situation here. It is a first order differential equation for v! All we need is to "just integrate[1] once."

$$\begin{aligned}
m\dot{v} &= -kmv \\
\frac{dv}{dt} &= -kv && \text{divide by } m\text{, use Leibniz notation} \\
\frac{dv}{v} &= -kdt && \text{multiply both sides by } dt \\
\frac{dv(t')}{v(t')} &= -kdt' && \text{change variable } t \to t' \\
\log v - \log v_0 &= -kt && \text{integrate from } t' = 0 \text{ to } t' = t \\
\log(v/v_0) &= -kt && \log(A/B) = \log A - \log B \\
v &= v_0 \exp(-kt) && \log X = Y \to X = \exp Y
\end{aligned}$$

This is a simple, but very important result.

$$v = v_0 \exp(-kt) \tag{2.3}$$

If the only force acting on the body is an air resistance that is linear in velocity, then the velocity will vanish exponentially.



Exponential decay

The equation of the form $dv/v = -kdt$ occurs ubiquitously in physics (and elsewhere). If v is not the velocity, but the number of a certain kind of particles with a finite lifetime $\sim 1/k$, (the symbol $\sim$ means "on the order of" or "up to a multiplicative constant") then this equation describes the population decay of that particle. As such, the Rutherford radioactive decay law $N(t) = N(0)\exp(-kt)$ can be understood with a similar microscopic mechanism: a radioactive particle has a certain probability to decay, kdt, in the small time interval, dt. Such a decay happens for naturally unstable particles (e.g. muons), or for virtually all excitations inside materials, e.g. the electron-hole excitation in solar cells, although in the latter case, the decay is followed by emission of visible light, not γ rays!

Note that the dimension of k is $1/T$ as can be seen easily from $F = -kmv$. So it makes a perfect mathematical sense that kt appears together in the exponential

[1] One comment on notation: log means the natural logarithm. So, $dv/v = d\log v$.

function. In the argument of a transcendental function, only a dimension-less quantity can appear, such as kt here.

Distance and velocity

In an accelerated motion, we often ask the question about the relationship between the distance and the velocity. In this example, that relationship is obtained easily by eliminating t from our solution, $x - x_0 = \frac{v_0}{k}(1 - \exp(-kt)) = \frac{v_0}{k}(1 - v/v_0)$, that is:

$$k(x - x_0) = v_0 - v$$

Actually, we could have gotten this much more easily. The trick is to use

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

and eliminate t from the EOM (equation of motion) itself. If we do that, then the EOM becomes $\frac{dv}{dx}v = -kv$, i.e.

$$dv = -kdx$$

from which the above result $k(x - x_0) = v_0 - v$ can be derived quickly. As you can see, this trick is quite useful, when the question does not involve t.

Namely, in this problem $1/k$ is a **time scale**, which determines the decay behavior. k is called a **decay constant**.

x is easy to get, since the exponential is readily integrable.

$$x - x_0 = \frac{v_0}{-k}(\exp(-kt) - \exp(-k \cdot 0)) \qquad \because \int \exp(-kt')dt' = \exp(-kt)/-k \tag{2.4}$$

$$x = x_0 + \frac{v_0}{k}(1 - \exp(-kt)) \tag{2.5}$$

$$= x_\infty - \frac{v_0}{k}\exp(-kt) \qquad x_\infty \overset{def}{=} x_0 + v_0/k \tag{2.6}$$

As the result of the air resistance, the range of motion is limited to a finite interval, and the particle converges to x_∞ never reaching it. The finite range of motion, $x_\infty - x_0 = v_0/k$, can be understood easily this way also. Due to the fractional decrease in the velocity $dv/v = -kdt$, the distance travelled during each small time interval dt

is slightly less than the distance travelled during the previous interval of the same duration, by the factor $1-\varepsilon$ where $\varepsilon = dv/v$. The total distance travelled is then given as a simple geometric series: $v_0 dt[1+(1-\varepsilon)+(1-\varepsilon)^2+\cdots]$ where $\varepsilon = dv/v = kdt$. This is the range of motion $= v_0 dt/(1-(1-\varepsilon)) = v_0 dt/\varepsilon = v_0 dt/(kdt) = v_0/k$.

This result can be described succinctly: **because the linear air resistance imposes a finite time scale $1/k$, the range of motion is limited to $\sim v_0/k$.** It just so happens that in this problem the $\sim$ sign ("on the order of" or "up to a multiplicative constant") can be replaced with the = sign.

Checks: Two symbols for the initial condition: x_0 and v_0. Notice that the dimension of v_0/k is $(L/T)T = L$, so our solution makes sense dimensionally, as it should. In the limit of $k \to 0$ we should recover $x = x_0 + v_0 t$. This is easily verified since $1-\exp(-kt) \approx 1-(1-kt) = kt$ when $|kt| \ll 1$. In the opposite limit $k \to \infty$, we would expect that the particle will be instantly stuck. Indeed, we see that all terms vanish except x_0, giving $x = x_0$.