Appendix C

Green's function for SHO

This is an optional topic—the method of Green's function for solving the SHO problem—highly recommended for your reading, if you have time. Specifically, this shows how one can obtain a particular solution for any given driving force. The contents of this chapter are formatted like a homework problem and contain all major results within this method. To skim the setup of the problem and the results, one can only read numbered equations.

Consider a SHO (simple harmonic oscillator) driven by an external force. Assume that $f(t) = F(t)/m$ is an arbitrary function.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x \ = \ f(t). \tag{C.1}$$

Here, $\beta \geq 0$, able to represent any physical cases (free, underdamped, critically damped and overdamped SHO). (As usual) we will be concerned with the particular solution only. In this problem, we will be looking for a particular solution that satisfies the following boundary condition

$$x(-\infty) = \dot{x}(-\infty) = 0. \tag{C.2}$$

That is, the system is "quiet" at the far end of the past, before any $f(t)$ is "turned on."

(a) Let us write $x(t) = \exp(\alpha_1 t)g(t)$ where α_1 is one of the solutions for the characteristic equation for the above ODE (ordinary differential equation):

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0.$$

Find the ODE satisfied by $g(t)$. It should be directly integrable over time. Do the integration from $-\infty$ to t, and use the above boundary condition, to show that $g(t)$ satisfies the following ODE, where α_2 is the other solution (which could be identical with α_1) of the above characteristic equation.

$$\dot{g} + (\alpha_1 - \alpha_2)g = \int_{-\infty}^{t} ds f(s) e^{-\alpha_1 s}.$$

<u>SOLUTION</u> ✎

$$\begin{aligned}
\dot{x}_p &= \dot{g}e^{\alpha_1 t} + \alpha_1 x_p \\
\ddot{x}_p &= \ddot{g}e^{\alpha_1 t} + 2\alpha_1 \dot{g} e^{\alpha_1 t} + \alpha_1^2 x_p \\
\ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= \left[\ddot{g} + (2\alpha_1 + 2\beta)\dot{g}\right] e^{\alpha_1 t} \\
\alpha_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\
\alpha_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \\
2\alpha_1 + 2\beta &= 2\sqrt{\cdots} = \alpha_1 - \alpha_2 \\
\ddot{g} + (\alpha_1 - \alpha_2)\dot{g} &= f(t) e^{-\alpha_1 t} \\
\dot{g} + (\alpha_1 - \alpha_2) g &= \int_{-\infty}^{t} ds f(s) e^{-\alpha_1 s} \qquad \text{Use } g(-\infty) = \dot{g}(-\infty) = 0
\end{aligned}$$

In the last step, we used the fact that $g(-\infty) = \exp(\alpha_1 \infty) x(-\infty) = 0$ and similarly for $\dot{g}(-\infty) = 0$. The key thing to note here is that, if $\beta > 0$, then α_1 (and α_2) is either a negative real number or a negative real number plus a complex number, so that $\exp(\alpha_1 \infty) = 0$. If $\beta = 0$, then α_1 (and α_2) is pure imaginary. In this case, $\exp(\alpha_1 \infty)$ is always a finite number, and $g(-\infty) = 0$ by virtue of $x(-\infty) = 0$. Similar arguments can be made to prove that $\dot{g}(-\infty) = 0$. ✎

(b) Now, define $h(t)$ by

$$g(t) = h(t) e^{(\alpha_2 - \alpha_1)t}$$

and show that the ODE for $h(t)$ is directly integrable. Show that $h(t)$ is given by

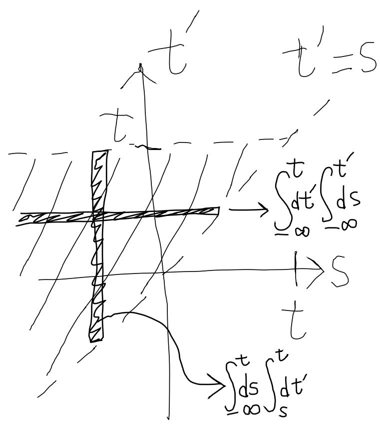
$$h(t) = \int_{-\infty}^{t} ds \int_{s}^{t} dt' e^{(\alpha_1 - \alpha_2)t'} e^{-\alpha_1 s} f(s).$$

[Hint: You need to use $\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} ds \cdots = \int_{-\infty}^{t} ds \int_{s}^{t} dt' \cdots$. Draw a diagram to show that this is true.]

Solution ✎ $\dot{g} + (\alpha_1 - \alpha_2)g = \dot{h}e^{(\alpha_2-\alpha_1)t}$

$$\begin{aligned} \dot{h}e^{(\alpha_2-\alpha_1)t} &= \int_{-\infty}^{t} ds f(s) e^{-\alpha_1 s} \\ h(t) &= \int_{-\infty}^{t} dt' e^{(\alpha_1-\alpha_2)t'} \int_{-\infty}^{t'} ds f(s) e^{-\alpha_1 s} \\ &= \int_{-\infty}^{t} ds \int_{s}^{t} dt' e^{(\alpha_1-\alpha_2)t'} e^{-\alpha_1 s} f(s) \end{aligned}$$

On the left hand side, we used the fact that $h(-\infty) = 0$. Since $h(t) = \exp(-\alpha_2 t)x(t)$, $h(-\infty) = 0$ for the same reason why $g(-\infty) = 0$ in (a). The change of the double integration range is explained in the diagram below.



It is important to note that $t' > s$ in the integration. ✎

(c) Show that $x(t)$ can be written as

$$x(t) = \int_{-\infty}^{t} dt'\, G(t-t') f(t'), \qquad \text{(C.3)}$$

where

$$G(t-t') = \theta(t-t') \int_0^{t-t'} du\, e^{(\alpha_1-\alpha_2)u}\, e^{\alpha_2(t-t')}, \qquad \text{(C.4)}$$

and $\theta(t-t')$ is the Heaviside step function (1 if $t > t'$ and 0 if $t < t'$).

<u>SOLUTION</u> ✎ We defined $x = \exp(\alpha_1 t)g$ and $g = \exp(\alpha_2 t - \alpha_1 t)h$. And so, $x = \exp(\alpha_2 t)h$. Using this, and also changing the integration variable $t' \to u = t' - s > 0$:

$$\begin{aligned} x(t) &= e^{\alpha_2 t}\int_{-\infty}^{t} ds \int_0^{t-s} du\, e^{(\alpha_1-\alpha_2)u} e^{-\alpha_2 s} f(s) \\ &= \int_{-\infty}^{t} ds \left[\int_0^{t-s} du\, e^{(\alpha_1-\alpha_2)u} e^{\alpha_2(t-s)}\right] f(s) \\ &= \int_{-\infty}^{t} dt'\, G(t-t') f(t') \qquad \text{Change variable } s \to t'. \end{aligned}$$

where

$$G(t-t') = \theta(t-t')\int_0^{t-t'} du\, e^{(\alpha_1-\alpha_2)u}\, e^{\alpha_2(t-t')}$$

and the step function comes in since in the integral $t > s$. ✏

(d) Show that if $f(t) = \delta(t-t_0)$ (the Dirac delta function—please look it up if you are not familiar with it), then

$$x(t) = G(t-t_0)$$

Such a function—the response of a system upon a unit delta function impulse—is generally called the **Green's function**. It is a concept that permeates throughout all physics. [Note: $f(t)$ should have the physical dimension of force/mass, different from 1/time. So, what does it mean that $f(t) = \delta(t-t_0)$? We assume that we have chosen the units so that the strength (i.e. the integral) of $f(t)$ is 1.]

<u>SOLUTION</u> ✎ The property of the delta function is that

$$\int \delta(t-t_0) F(t) dt = F(t_0)$$

as long as the integration range covers both sides of t_0, no matter how small the integration range is. Thus, it follows that when $f(t) = \delta(t-t_0)$, $x(t) = \int_{-\infty}^{t} dt'\, G(t-t')\delta(t-t_0) = G(t-t_0)$ as long as $t > t_0$. Since $G(t-t_0)$ is already zero when $t < t_0$, we can see that $G(t-t_0)$ is the solution at all times. [At $t = t_0$, $G \to 0$, and there is no ambiguity of assigning the value of G at $t = t_0$.] ✏

(e) Show that **for an under-damped or free SHO**

$$G(t-t') = \frac{\theta(t-t')}{\omega_1} e^{-\beta(t-t')} \sin(\omega_1(t-t')) \tag{C.5}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

SOLUTION ✎

$$\int_0^{t-t'} du\, e^{(\alpha_1-\alpha_2)u}\, e^{\alpha_2(t-t')} = \frac{e^{(\alpha_1-\alpha_2)(t-t')}-1}{\alpha_1-\alpha_2}\, e^{\alpha_2(t-t')}$$

`In this case,` $\alpha_1 = -\beta + i\omega_1$ `and` $\alpha_2 = -\beta - i\omega_1$`.` $\alpha_1 - \alpha_2 = 2i\omega_1$`. So, the above becomes`

$$\frac{e^{2i\omega_1(t-t')}-1}{2i\omega_1}\, e^{-i\omega_1(t-t')}\, e^{-\beta(t-t')}$$

`which is, using` $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$`,`

$$\frac{1}{\omega_1} e^{-\beta(t-t')} \sin(\omega_1(t-t'))$$

`According to (c), the Green's function is this times` $\theta(t-t')$`, so we get the answer.` ✐

(f) Show that **for a critically damped SHO**

$$G(t-t') = \theta(t-t')(t-t')\exp(-\beta(t-t')). \tag{C.6}$$

SOLUTION ✎ `In this case,` $\alpha_1 - \alpha_2 = 0$ `and` $\alpha_2 = -\beta$`. So,`

$$\int_0^{t-t'} du\, e^{(\alpha_1-\alpha_2)u}\, e^{\alpha_2(t-t')} = \int_0^{t-t'} du\, e^{-\beta(t-t')} = (t-t')e^{-\beta(t-t')}$$

✐

(g) Show that **for an over-damped SHO**

$$G(t-t') = \frac{\theta(t-t')}{\gamma} e^{-\beta(t-t')} \sinh(\gamma(t-t')) \tag{C.7}$$

where $\gamma = \sqrt{\beta^2 - \omega_0^2}$.

SOLUTION ✎ `This part is similar to part (e), except that now` $\alpha_1 = -\beta + \gamma$ `and` $\alpha_2 = -\beta - \gamma$`.` $\alpha_1 - \alpha_2 = 2\gamma$`. Doing similarly as in (e), but using` $\sinh(x) = \frac{e^x - e^{-x}}{2}$`, we get the result.` ✐

Using the above results for $f(t) = f_0 e^{i\omega t}$ one can re-derive all standard results for $x(t)$ for a sinusoidal force. What is more: we can directly handle *any* function $f(t)$!