Due Dec. 4, Thursday.
The perfect score of this homework is 120 points.

Problem 1 (20 + 10 points) Consider a cylindrical shell with uniform density and infinite length. The cylinder parallel to the z direction. Mass exists between radii a and b ($a < b$) while no mass exists for $\rho < a$ and $\rho > b$, where $\rho = \sqrt{x^2 + y^2}$. Let the mass per unit length along the cylinder be A.

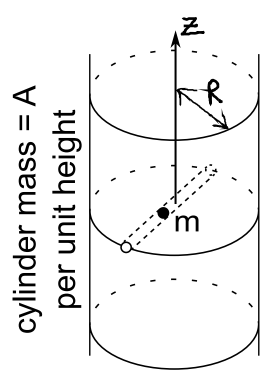
(a) Find the field $\vec{g}$ due to the cylinder at any position (including inside the cylinder). [Use the Gauss law.]

(b) Find the potential Φ due to the cylinder. In the cylindrical coordinate system (ρ, ϕ, z), sketch Φ as a function of ρ starting from 0 to a value larger than b.

(c) Now consider the situation where there are two of these cylinders separated by distance $2D$ with $D > b$. Clearly the midpoint between the two cylinders is an equilibrium point. Find the nature of this equilibrium point (stable, unstable, saddle) by examining the behavior of Φ near that point.

(d) (Extra credit 10 points) Suppose that *four* of these cylinders are placed centered at four corners of a square, whose diagonal is $2D$. Each cylinder remains parallel to the z axis, while the square is in the xy plane. Find the nature of the equilibrium point at the center of the square.

Problem 2 (30 points) An infinitely long cylinder has mass A per unit length (or height; see below).



This problem is cumulative: assumptions made in one part apply to parts later to it, but not to parts prior to it.

(a) Find the gravitational field $\vec{g}$ due to this cylinder, inside and outside the cylinder.

(b) Find the gravitational potential Φ corresponding to $\vec{g}$, inside and outside the cylinder.

(c) A small hole is drilled as pictured above, perpendicular to the z axis of the cylinder. Assume that there is no friction between the hole and the mass m. (i) Show that the motion of a test mass m inside the hole is a simple harmonic motion. (ii) Find the time it would take for the mass m to travel all the way to the other side of the hole when it is dropped with zero speed into the hole.

(d) Additionally, the whole system is rotating at an angular speed ω_0 around the z axis. Find the effective potential of this problem. Assume that m remains inside the hole thanks to appropriate (but unspecified) initial conditions.

(e) The origin—i.e., the point where the z axis intersects the hole path—is an equilibrium point of this system. Is there a critical angular speed above which it becomes an unstable equilibrium? If so, find it. If not, prove that the origin is always a stable equilibrium point.

(f) Find the normal force that the surface of the hole exerts on mass m as a function of the position and the velocity.

Problem 3 (10 points) Assume that the Earth's orbit around the Sun is a circle. Suddenly, the Sun's mass decreases by half, and the Earth's mass increases two-fold. The Earth's position and velocity remain unchanged while these sudden changes occur. What is the new orbit of the Earth? Will the Earth escape the Sun? [Hint: Start by considering the relations between E, T, and U for a circular motion.]

Problem 4 (15 points) A spacecraft is parked in a circular orbit 600 km above Earth's surface. We want to use a Hohmann transfer to send the spacecraft to the Moon's orbit. What are the total Δv and the transfer time required?

Problem 5 (15 points) Two stars, with masses m_1 and m_2, are circling each other under the influence of the gravitational force. The period of the motion is τ and the radius of the circle is R. Assume that, suddenly, the two stars lose their speeds completely. Subsequently, the gravitional force causes them to fall towards each other and crash.

(a) What are the speeds of the two stars (in terms of m_1, m_2, R), when the distance between the two stars is halved?

(b) Show that it takes $\frac{\tau}{4\sqrt{2}}$ before the stars crash. Assume that the radii of the stars are much smaller than R, and thus negligible.

Problem 6 (30 points) A mass m is making a circular orbit around a very heavy mass $M (\gg m)$, due to the gravitational force by M. There is another mass $m_2 = 2m$, which is also moving in the gravitational field of mass M, in the same plane of motion as m. It is observed that the two small masses (m and m_2) collide head-on. Just before the head-on collision, m has velocity $\vec{v}$, while m_2 (= $2m$) has velocity $-2\vec{v}$. Ignore any gravitational interaction between m and m_2; this means that the two masses do not interact except during the collision. Assume that the collision is *elastic*.

(a) What is the shape of the orbit for mass m_2 (= $2m$) prior to the collision? Explain quantitatively.

(b) Find the velocities of m and m_2 right after the collision.

(c) What are the shapes of the orbits for mass m_2 and m after the collision? Explain quantitatively.

Problem 7 (30 points) A circular disk of total mass M and radius R is mounted horizontally. It is mounted in such a way that it is free to rotate horizontally around a vertical pole that fits tightly in a small hole at the center of the disk. The rotational inertia for the disk is $\frac{1}{2}MR^2$ to a good approximation, assuming that the central hole is small. A monkey of mass m runs and lands on the edge of the disk with a horizontal velocity. Ignore the size of the monkey relative to the radius R.

(a) The event of the monkey landing on the disk is a collision. Is the total angular momentum conserved during this collision? Is the total kinetic energy conserved (i.e. is it an elastic collision)? Is the total linear momentum conserved? For each answer, briefly explain why (in terms of torque/work/force or any Newtonian concepts).

(b) Assume that the monkey's initial velocity is $v_r\hat{r} + v_\theta\hat{\theta}$, where r, θ define the polar coordinate system of the disk. What is the angular velocity of the subsequent motion?

(c) In this subsequent motion, sketch the trajectory of the center of mass of the disk+monkey system. Identify the external force that is responsible for this motion of the center of mass.

(d) Now, let us assume that the disk is mounted vertically. The surface gravity is g, downward. The monkey falls down vertically and lands on the disk. For this collision, answer the same set of questions of (a). As usual, assume that the time of collision is very short.

(e) In this vertical setup, assume that the monkey's speed just before it lands on the disk is v. Let the horizontal distance of the monkey from the center of the disk prior to the collision be b (so this is just like the impact parameter). Find the minimum value of v, as a function of m, M, R and b, for which in the subsequent motion the sign of $\dot{\theta}$ does not change (i.e. the motion is a rotation instead of an oscillation).