Due 1 PM (sharp), Nov. 21, Friday. Submit your work at ISB 249.

Problem 1 (30 points) A bead is constrained to move along a solid wire whose shape is given by

$$z = \frac{1}{2a}x^2 + \frac{1}{4a^3}x^4$$

where x and z are the coordinates of the xz plane where the wire resides. There is no friction. Here, a is a length scale, z is the vertical coordinate, and the surface gravity is defined by the constant gravitational field $-g\hat{z}$. The wire (and thus the xz plane) is being rotated at a constant angular frequency ω around the z axis.

(a) Find the Lagrangian, $L(x, \dot{x}, t)$.

(b) Find the canonical momentum, p_x, and the Hamiltonian H. Define the effective mass *function* $m_{eff}(x)$ such that $p_x = m_{eff}\dot{x}$. Is H conserved?

(c) Find the effective potential $U_{eff}(x)$ such that $H \equiv T_{eff} + U_{eff}$, where $T_{eff} = \frac{1}{2}m_{eff}\dot{x}^2$.

(d) Find all equilibrium points of the motion by solving $dU_{eff}/dx = 0$. Why does this method work even if m_{eff} is a *function* of x?

(e) Show that there is a critical frequency ω_c such that if $\omega < \omega_c$ then there is only one stable equilibrium point, while if $\omega > \omega_c$ then there are three equilibrium points of which two are stable and one is unstable. Find ω_c.

(f) Find the frequency for small oscillation around each stable equilibrium point for $\omega < \omega_c$ and $\omega > \omega_c$. Show that the frequency goes to zero, as $\omega \to \omega_c$ in each of these cases.

Problem 2 (30 points) **Rolling without slipping.** A slab of mass M is sliding without friction on a slope with angle α with respect to the horizontal. On the slope there is a small opening through which the top of a wheel mounted in a cavity underground is exposed. The wheel is free to rotate around its axle without any friction. As the slab comes down on the slop, it makes contact with the wheel, the wheel rotates without slipping with respect to the slab. The wheel's mass is m, its radius is R and its rotational inertia is given by $\gamma m R^2$, where γ is a number of order 1, determined by the mass distribution within the wheel. In this problem, we consider the system (the slab and the wheel) only while they are in contact.

(a) Find the acceleration of the slab, as a function of g, α, M, m and γ. Find the value of the acceleration in the limits of $m \gg M$ and $M \gg m$ and show that your answer is reasonable in those limits.

(b) Is the energy conserved in this system? Explain as briefly as you can.

(c) Find the force of constraint at the contact between the slab and the wheel. Identify the nature of this force. Calculate the power delivered to the slab by this force and show that it is not zero. Discuss your finding here in detailed relation to the answer of the previous part and remove any doubt that any of your answers is incorrect.

Problem 3 (20 points; extra credit) **Poisson bracket formalism of Classical Mechanics.** From the Hamiltonian formalism, the Poisson bracket of two general functions $g(p_i, q_i, t)$ and $h(p_i, q_i, t)$ is defined as

$$[g, h] = \sum_i \left(\frac{\partial g}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q_i} \right)$$

and we can construct yet another formalism of Classical Mechanics (see (c) below). Here, i is the index for generalized coordinates. Assuming that there are n generalized coordinates, g and h are functions of $2n+1$ independent variables: $p_1, \cdots, p_n, q_1, \cdots, q_n, t$.

(a) Prove that $[g, h] = -[h, g]$, $[g, g] = 0$.

(b) Prove that $[q_i, p_j] = \delta_{ij}$ (Kronecker delta).

(c) Prove that $\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}$.

(d) Calculate $[\vec{L}_i, H]$ explicitly, where $\vec{L}_i$ and H are from problem 4 of homework 5 (parts a-c), and show that it is zero, consistent with what we just established in the previous part. [Hint: item 5 in page 2 of LN 4 may be useful.]

(e) Calculate $[\vec{P}_i, H]$ explicitly, where $\vec{P}_i$ and H are from problem 4 of homework 5 (parts a-c). Confirm that it is *not* zero, in general.

[Note: The Poisson bracket is to Classical Mechanics, what the "commutator" is to Quantum Mechanics (QM). Their properties are very similar. In QM, the definition of the commutator is $[A, B] = AB - BA$, where A, B are quantum mechanical "operators" (matrices, basically). H, q_i, p_i etc. become operators in QM. All of the above properties have direct analogues in QM commutators (the only difference being the multiplicative constant $i\hbar$ appearing here and there in QM). In particular, the QM version of (b) is the canonical quantization condition, responsible for the Heisenberg uncertainty principle. The QM version of (c) is the Heisenberg equation of motion, completely equivalent to the Schrödinger equation of motion.]