Due Nov. 13, Thursday.

The perfect score of this homework is regarded as 90 points. However, the maximum score that you can get is 120 points, and so there is a significant amount of extra credit (30 points). You can decide which problems to do and which problems to do partially or to skip. I recommend that you do all problems; at least you should think hard about all problems and discuss your thoughts with fellow students.

Problem 1 (20 points) **Fermat's principle and the principle of least action.** Let us consider a photon, the quantum of light, with angular frequency ω. While the study of photons is definitely not a realm of classical mechanics, some part of it can be understood using classical mechanics. The famous "Fermat's principle" of geometric optics is one of them. This principle states that "the light travels in a path that minimizes the time spent." This law, just like the principle of least action (PoLA), need to be re-stated to be valid in general cases. This is done in (a). Then, we study a simple case, in parts (b,c), where Fermat's principle in its old form can be derived from the PoLA.

(a) Consider a photon of angular frequency ω moving in a medium characterized by a position dependent refractive index $n(\vec{r})$. Write down the principle of least action, as we defined it in class, applied to this problem. Show that it means that the integral $\int_1^2 n(\vec{r})ds$ is stationary for the true path, where s measures the distance traveled.

[Some potentially helpful remarks are offered now. At this point, you should not think about Fermat's principle at all. Just think PoLA. To figure out the Lagrangian[1], note that the kinetic energy of a photon is given as (from your "modern physics" type of course)

$$K = pv = \hbar k v$$

where k is the wave number $(2\pi/\lambda)$, and the potential energy of a photon is zero. Here, $\hbar$ is the Planck constant, $h/(2\pi)$, v is the speed of light in the medium:

$$v = \frac{c}{n}$$

where c is the speed of light in vacuum. So, v is a position dependent function. This means k is position dependent as well, since

$$kv = \text{position independent}$$

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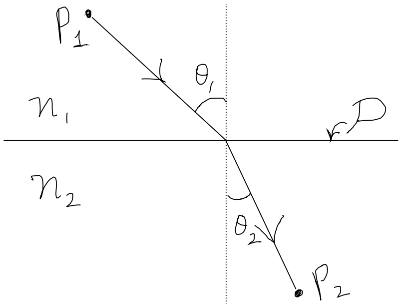
[1] From the vantage point of view of modern field theory, one must note that it is actually *incorrect* to apply $L = K - U$ to a relativistic case such as this. The Lagrangian in the relativistic mechanics or for the electromagnetic field must be obtained in a more general way based on symmetry and other consideration (see, e.g., Landau and Lifshitz, "The Classical Theory of Fields"). Here, it turns out that the current heuristic approach gives rise to the correct minimization principle. The action that we get is the so-called abbreviated action that is relevant for the Maupertuis principle—the "original" least action principle that predates Hamilton's least action principle (which is what we learned in our class).

is equal to its value in vacuum (due to the energy conservation).]

(b) Consider a simple case when the space is divided into two parts by a boundary plane ($\mathcal{P}$). One side of plane $\mathcal{P}$ has a constant refractive index n_1 and the other side another constant refractive index n_2. Consider two arbitrary points P_1 and P_2, one in the n_1 region and the other in the n_2 region. It follows that the plane that contains P_1, P_2, and is perpendicular to $\mathcal{P}$ is uniquely determined. Within this plane, the following diagram can be drawn. Derive, by *minimizing* the action obtained in (a), Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Your solution should show your analysis of this problem with respect to all variational paths in three dimensions.



However, this is *not* to be a complicated calculus of variation problem. First of all, you can take it for granted (based on the intuition) that the shortest distance between two points in space (that is, the Euclidean space) is achieved by a straight line. Consider a point P_3, which is an arbitrary point in the plane $\mathcal{P}$ and which the path of the photon goes through. Then, all you need to do is to analyze the behavior of the action S as a function of P_3.

(c) Show that the function that is minimized in (b) is the total time of the motion, up to a multiplicative constant.

Problem 2 (15 points) **The *brachistochrone* problem and the generalized coordinate.** The so-called **cycloid** is a curve that is generated by a point of a rolling wheel. Assume that you are in a room with a ceiling whose vertical crossection is a cycloid shape:

$$\begin{aligned} x_c &= a(\phi + \sin\phi), \\ z_c &= a(1 - \cos\phi). \end{aligned}$$

Here ϕ is a real number, and z (x) is the vertical (horizontal) coordinate. Consider a pendulum consisting of a thin string of fixed length $4a$, attached to the ceiling at the position $a\pi\hat{x}+2a\hat{z}$ (corresponding to $\phi = \pi$), and a mass m attached at the other end of the string. When this pendulum is allowed to oscillate in the xz plane, one can show, using a bit of calculus and trigonometry, that the resulting position (x, z) of the mass follows the path of a cycloid as well (you will be given an extra credit, if you actually prove this):

$$\begin{aligned} x &= a(\phi - \sin\phi), \\ z &= a(\cos\phi - 1). \end{aligned}$$

Now, ϕ should be thought of as a dynamical variable that describes the pendulum motion. The stable equilibrium position corresponds to $\phi = \pi$. For a given energy value, the maximum value of $|\phi - \pi|$ can be defined as the "amplitude" of the oscillation. The amplitude has this physical constraint being $\le \pi$.

(a) Write down the Lagrangian for this pendulum motion.

(b) Re-express the Lagrangian in terms of the generalized coordinate $q \equiv \cos(\phi/2)$.

(c) Use the Lagrange equation of motion for q, to prove that the period of the oscillation *is independent of the amplitude* of the oscillation, and is given by $4\pi\sqrt{a/g}$.

[Note: The cycloid is also the solution of the *brachistochrone* problem. In this problem, one drops a bead from rest on a frictionless curve. The bead travels down under the influence of the gravity on the curve. The question posed is "what is the shape of the curve that minimizes the time of travel from the origin to a given fixed final point?".]

Problem 3 (15 points) **Mechanical similarity.** The scaling of the Lagrangian

$$L'(q, \dot{q}, t) = C L(q, \dot{q}, t)$$

where C is a non-zero constant, does not change physics, in the sense that the Hamilton's principle $\delta S' = 0$ is completely equivalent to $\delta S = 0$.

(a) Assume a power law potential energy $U = ax^n$ for a one dimensional motion. Suppose in the Lagrangian we scale $t \to Bt$ and $x \to B^\alpha x$. Find the exponent α (in terms of n) that guarantees that the new Lagrangian describes the same physical system.

(b) For the $n = 2$ case, show that your answer is consistent with the fact that the period of a simple harmonic motion is independent of the amplitude. (Note that, in this case, it may be better to consider the similarity transformation as: $x \to Dx$ and $t \to D^{1/\alpha}t$, with $D \overset{def}{=} B^\alpha$.)

(c) For the case of $n = 4$, we expect to have a bound oscillatory motion as well. In this case, will the period (τ) depend on the amplitude A? If so, how (i.e., what is β in $\tau \propto A^\beta$)?

(d) Now, consider a three dimensional problem with a potential that behaves as

$$U(\vec{r}) = a\, r^n$$

where $r = |\vec{r}|$. Use the same argument ($t \to Bt$, $\vec{r} \to B^\alpha \vec{r}$, $L \to CL$), and obtain the α exponent.

(e) For the Kepler problem, describing the motion of a planet around the Sun, $n = -1$. Show that your answer is consistent with Kepler's third law, which says that the period of a planetary orbit is proportional to $a^{3/2}$, where a is the linear scale of the orbit, e.g., the semi-major axis (ellipse) or the radius (circle).

Problem 4 (20 points) **Two body problem.** We consider a two body problem with a *central force*, whose Lagrangian is given by

$$L = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

where two bodies with masses m_1 and m_2 interact via a *central potential* $U(|\vec{r}_1 - \vec{r}_2|)$. A central potential is a potential that depends only on the *distance.*

(a) Use the following transformation to re-express the Lagrangian in terms of $\vec{R}$, $\dot{\vec{R}}$, $\vec{r}$, $\dot{\vec{r}}$, M and m.

$$\vec{R} \overset{def}{=} \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{r} \overset{def}{=} \vec{r}_1 - \vec{r}_2$$

$$M \overset{def}{=} m_1 + m_2$$

$$\frac{1}{m} \overset{def}{=} \frac{1}{m_1} + \frac{1}{m_2}$$

Here, $\vec{R}$ is the position vector of the center of mass and $\vec{r}$ is the relative position vector. M is the total mass, and m is the "reduced mass" (often called μ). Your answer should show explicitly that $\vec{R}$ and $\vec{r}$ are completely decoupled in the sense that

$$L = L_{cm}(\vec{R}, \dot{\vec{R}}) + L_i(\vec{r}, \dot{\vec{r}})$$

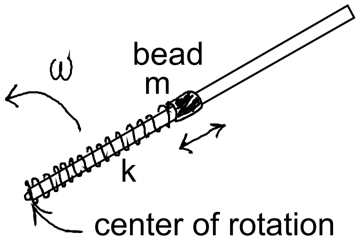
where L_{cm} is the part of the Lagrangian that depends on $\vec{R}$ and its time derivative, and L_i is the part of the total Lagrangian that depends on $\vec{r}$ and its time derivative. L_{cm} is the Lagrangian for the *center of mass* degrees of freedom $\vec{R}$, and L_i for the *relative (or internal)* degrees of freedom $\vec{r}$.

(b) Express the total momentum $\vec{P} \overset{def}{=} m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2$ in terms of the momenta associated with L_{cm} and L_i. Likewise, express the total angular momentum $\vec{L} \overset{def}{=} m_1\vec{r}_1 \times \dot{\vec{r}}_1 + m_2\vec{r}_2 \times \dot{\vec{r}}_2$ in terms of the angular momenta associated with L_{cm} and L_i. Discuss how your results make sense, using an example that consists of two particles interacting by a central potential, e.g. a binary star system.

(c) Which of these quantities are conserved and why? H (Hamiltonian), E (energy), $\vec{P}$ (total momentum), $\vec{P}_{cm}$ (center of mass momentum associated with L_{cm}), $\vec{P}_i$ (internal momentum associated with L_i), $\vec{L}$ (total angular momentum), $\vec{L}_{cm}$ (center of mass angular momentum), $\vec{L}_i$ (internal angular momentum).

(d) Consider a scenario that the potential is replaced by a more general form, $U(\vec{r}_1 - \vec{r}_2)$. How then do the answers to the previous part change? Discuss how likely this scenario is.

Problem 5 (30 points) A bead is constrained to move, without friction, along a solid wire. The bead is connected to a spring with the spring constant k and the equilibrium length l. The spring wraps tightly around the wire, without any friction between the spring and the wire. The wire begins to rotate by a certain external force, with the other end of the spring fixed at the center of rotation, together with the end of the wire. The rotational motion of the wire reaches a constant angular velocity ω. We are interested in the motion *only after* this constant angular velocity is reached.



(a) Find the Lagrangian in terms of x, $\dot{x}$ and t, where x is the displacement of the spring relative to l.

(b) Find the canonical momentum, p, corresponding to x. What is the meaning of it?

(c) Find the Hamiltonian, H. Is it equal to the energy $E = K + U$?

(d) Is the Hamiltonian conserved? Explain your answer in one sentence. Is the energy conserved? Explain your answer.

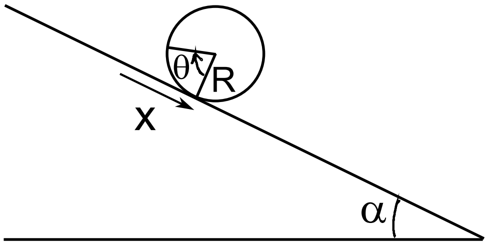
(e) By looking at H, identify the effective potential, U_{eff}, such that $H = \frac{1}{2}m\dot{x}^2 + U_{eff}(q)$.

(f) Now, consider ω as a parameter that can be varied from experiment to experiment. Show that there is a "critical" value of ω_c such that, a stable equilibrium exists only when $\omega < \omega_c$. Find ω_c, and find x_{eq}, the stable equilibrium point.

(g) Explain the Newtonian physics at the stable equilibrium point $x = x_{eq}$. Is there any other equilibrium point other than the stable equilibrium point?

(h) Find the angular frequency, ω_0, for small oscillation around the stable equilibrium point. How does the oscillation frequency behave as ω_c is approached? Keeping this and the equilibrium value of x in mind, sketch the potential energy as a function of x, when $\omega \ll \omega_c$ and $\omega \to \omega_c - 0^+$.

Problem 6 (20 points) An object with a circular crossection is rolling down on a slope, without slipping, as depicted here. In the figure below, x is the distance that the object travels on the slope, while θ is its angular displacement. The rotational inertia is given by $\gamma m R^2$, where m is the total mass and γ is an $O(1)$ constant, which depends on the details (shape and mass distribution) of the object ($\gamma = 1/2$ for a uniform solid cylinder, or 1 for a hollow cylinder, etc.) but not on m or R.



(a) Find the Lagrangian L. In this problem, you are required to express L as a function of $q, \dot{q}, t$, with *one* generalized coordinate q, which can be either x or θ. You must investigate and use the fact that $\theta = \theta(x)$ and $x = x(\theta)$.

(b) Find the canonical momentum, p, corresponding to the generalized coordinate that you defined. Discuss the meaning of the canonical momentum.

(c) Find the Hamiltonian, H. Is it equal to the energy $E = K + U$? Is it conserved?

(d) Find the equation of motion, and find the acceleration $\ddot{x}$ down the slope.

(e) Is there a friction force involved in this problem? Discuss the implication of your answer to your answer for (c).