Due Oct. 30, Thursday.

Problem 1 (10 points) For each of the following statements, prove it if it is true, or provide one counter-example if it is not true. [*Note:* In this problem, we are considering functions of x and operators on those functions. If you like you can consider functions of t. That is, you can change all x's to t's below, if you like. Changing x to t is more in line with Newton's equation (then $f(t)$ can be $x(t)$ or $v(t)$).]

(a) $Lf(x) = 3$. L is a linear operator.

(b) if L is a linear operator, then L^n for any $n = 0, 1, 2, \cdots$ is a linear operator. [Note that L^n defines a repeated application of L (if $n > 1$), or no application at all ($n = 0$).]

(c) If L_1, L_2 are linear operators, then $\alpha L_1 - \beta L_2$ is also a linear operator, for any constants α, β. [Note: $(L_1 - L_2)f(x) \overset{def}{=\!=} L_1 f(x) - L_2 f(x)$ and $(\alpha L)f(x) \overset{def}{=\!=} \alpha g(x)$ where $g(x) = Lf(x)$.]

(d) If L is a linear operator, then $\exp(L)$ is also a linear operator. Note: what is $\exp(L)$? It is $\sum_n \frac{L^n}{n!}$.

(e) $Lf(x) = \log f(x)$. L is a linear operator.

(f) $Lf(x) = f(x)^3$. L is a linear operator.

(g) $\cos\left(i\frac{d}{dx}\right)$ is a linear operator. Note: what is $\cos\left(i\frac{d}{dx}\right)$? It is $\sum_{n=even} \frac{1}{n!}\frac{d^n}{dx^n}$.

Problem 2 (20 points) Consider the following potential energy.

$$U(x) = a\left(\frac{b^{12}}{x^{12}} - 2\,\frac{b^6}{x^6}\right)$$

where $a, b > 0$ are constants and $x > 0$.

(a) Find all equilibrium points and discuss their stability.

(b) Make a sketch of $U(x)$, noting the values at any equilibrium point and the limits of $x \to 0^+$ and $x \to \infty$.

(c) What is the condition on the total mechanical energy E for a bound state motion to occur?

(d) What is the period T for the simple harmonic motion near the minimum energy (assume mass m)?

Problem 3 (40 points) Consider the simple harmonic oscillator (SHO) problem

$$m\ddot{x} = -kx - 2m\beta\dot{x}$$

and its general solution for under-damping ($\omega_0 > \beta \geq 0$)

$$x = A\exp(-\beta t)\cos(\omega_1 t + \theta_0)$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

(a) Find the expression for the kinetic energy K.

(b) Find the expression for the potential energy U.

(c) Find the expression for the total mechanical energy $E = K + U$ and make a sketch of $E(t)$ as a function of time, clearly noting the overall exponential decay time scale, the sinusoidal modulation period, and the overall sign of E.

(d) Differentiate the above expression to obtain dE/dt. Verify that this is what you would expect from $dE/dt = dW_{nc}/dt$, where dW_{nc} is the infinitesimal work done by the non-conservative force of this problem.

(e) **For the rest of this problem, assume $\beta \ll \omega_0$ and define $\langle Q\rangle$ as the time average of a quantity Q over one period $\tau = 2\pi/\omega_1$.** Find $\langle K\rangle$, $\langle U\rangle$, and $\langle E\rangle$. Note that the time average does not mean that $\langle Q\rangle$ is time independent. It is just coarse-grained.

(f) Look up the "virial theorem" for power law forces, state it, and verify that relationship between $\langle K\rangle$ and $\langle U\rangle$ is in agreement with the theorem.

(g) Show that the Q factor ($Q \equiv \frac{\sqrt{\omega_0^2 - 2\beta^2}}{2\beta}$) of this system can be written as

$$Q \approx 2\pi \frac{\langle E\rangle}{|\Delta E|}$$

where ΔE is the energy loss in one period τ.

Problem 4 (20 points) Consider a driven SHO with *no* damping.

$$\ddot{x} + \omega_0^2 x = A\cos(\omega t)$$

(a) Find the general solution of this equation when $\omega \neq \omega_0$. Identify the complementary function $x_c(t)$ and the particular solution $x_p(t)$, and discuss the time dependence of each.

(b) Find the general solution of this equation when $\omega = \omega_0$. Identify the complementary function $x_c(t)$ and the particular solution $x_p(t)$, and discuss the time dependence of each. Find the phase shift of $x_p(t)$. [Hint: assume $x(t) = g(t)\exp(i\omega t)$ for the particular solution of the complex-plane version of the above EOM and solve for $g(t)$.]