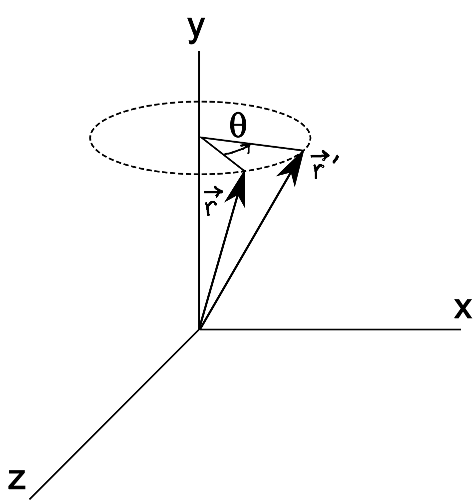
Due Oct. 16, Thursday.

Problem 1 (10 + 5 points) Consider a coordinate transformation by which a vector is rotated around the y axis by θ. Let us define the matrix $\overleftrightarrow{O}$ as

$$\vec{r}\,' = \overleftrightarrow{O}\,\vec{r}$$



(a) Find the matrix $\overleftrightarrow{O}$ as a function of θ. Do not assume that the matrix is an orthogonal matrix, prior to finding it, but do show briefly, after finding it, that it is an orthogonal matrix.

(b) Find all eigenvalues of $\overleftrightarrow{O}$. [Note: You need to find the determinant of a 3×3 matrix, and I trust that this topic was covered in your math/math-phys courses. The matrix at hand is "almost 2-dimensional" and so it should not be so bad.]

(c) (Extra credit 5 points; but you *must* read this one!) Prove that, in general, an eigenvalue, say λ, of an orthogonal matrix is a complex number of unit magnitude, $|\lambda| = 1$. [Make sure that your answer in part (b) agrees with this fact!] One important thing to keep in mind here is that the eigenvector of an orthogonal matrix is generally a complex vector. So, write down the eigenvalue equation for the matrix, and then consider its Hermitian conjugate, not its transpose, to start solving this problem. For a real matrix, such as an orthogonal matrix, its Hermitian conjugate is, of course, identical with its transpose. [Note: an orthogonal matrix is always diagonalizable, as it is a "normal matrix"—please look up what a normal matrix is, if you need to refresh your memory.]

Problem 2 (30 points) A particle is moving vertically under the influence of the air resistance $-mkv$, where the negative sign means "opposite to the velocity." It also experiences the downward gravitational force with magnitude mg, where g is a constant. The identity

$$a = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$$

may be useful for this problem.

(a) The particle is thrown upwards with initial speed v_0. Find the maximum height (h) of the particle as a function of v_0, k and the terminal speed, v_∞ (you must find v_∞), by considering the upward motion only.

(b) Show that h behaves as you expect (no air resistance behavior), in the limit of $\frac{v_0}{v_\infty} \to 0$.

(c) Show that $h \to \frac{v_0}{k}$ in the opposite limit of $\frac{v_0}{v_\infty} \to \infty$. Explain physically why this behavior occurs.

(d) Now, consider the downward motion, starting from zero velocity, and find the relation between the speed v and the vertical distance y that it travels down from the maximum height.

(e) Show that $y \to \frac{v^2}{2g}$ (as you expect) in the limit $\frac{v}{v_\infty} \to 0$.

(f) Find the limiting form of y after a long time has passed (and so the terminal velocity is approached; if need be, you can remove the ground so that the particle keeps going down). Explain physically why you get such a limiting form.

Problem 3 (15 points) Let us consider a vertical toss with initial upward speed v_0 just like in the previous problem, but with air resistance given by $-mkv^2$. Find the speed, v_f, when the particle comes back to the initial height. You must express v_f as a product of v_0 and a function of dimensionless parameter(s).

Problem 4 (20 + 5 points) Consider the previous problem, which we will solve using the perturbation only, ignoring the exact solution, for our training. So, in this problem, we will assume small air resistance, and solve the equation of motion perturbatively.

(a) (15 points) Find the leading order correction due to the air resistance for (i) the time that it takes for the particle to reach the maximum height, (ii) the maximum height, (iii) the time it takes for the particle to come back to the original vertical position, (iv) the speed when it comes back to the original vertical position.

(b) (5 points) Make a sketch of your answer for part (iv), along with the exact answer (problem 3). Compare the two curves and discuss.

(c) (Extra credit 5 points) Use the work-energy theorem to re-derive, and thus verify, your answer for part (iv) of (a).