Due Oct. 9, Thursday.
10 points for each problem, unless stated otherwise.

Problem 1 Assume that $\overleftrightarrow{O}_1$ and $\overleftrightarrow{O}_2$ are arbitrary orthogonal matrices of the same dimensions. For each of the following statements, prove it if it is true, or provide one counter-example (using 2×2 matrices is good enough) if it is not true.

(a) $\overleftrightarrow{O}_1 + \overleftrightarrow{O}_2$ is also orthogonal.

(b) $\overleftrightarrow{O}_1\overleftrightarrow{O}_2$ is also orthogonal.

(c) $r\overleftrightarrow{O}_1$ is also orthogonal, where r is an arbitrary real number.

(d) $\overleftrightarrow{O}_1^t$ is also orthogonal.

Problem 2 If $\vec{a}_1 = \lambda_1\hat{x} + \lambda_2\hat{y} + \lambda_3\hat{z}$, $\vec{a}_2 = \mu_1\hat{x} + \mu_2\hat{y} + \mu_3\hat{z}$, and $\vec{a}_3 = \nu_1\hat{x} + \nu_2\hat{y} + \nu_3\hat{z}$, show that

$$\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}.$$

Deduce that $\vec{a}_i \cdot (\vec{a}_j \times \vec{a}_k)$ remains the same for $(i, j, k) = (1, 2, 3), (3, 1, 2)$, or $(2, 3, 1)$.

Problem 3 For any matrices $\overleftrightarrow{A}$ and $\overleftrightarrow{B}$ for which the product $\overleftrightarrow{A}\overleftrightarrow{B}$ is well-defined, show that (a) $\left(\overleftrightarrow{A}\overleftrightarrow{B}\right)^t = \overleftrightarrow{B}^t\overleftrightarrow{A}^t$ and (b) $\left(\overleftrightarrow{A}\overleftrightarrow{B}\right)^{-1} = \overleftrightarrow{B}^{-1}\overleftrightarrow{A}^{-1}$. (c) With $\overleftrightarrow{A}$ and $\overleftrightarrow{B}$ physically interpreted (e.g. as rotations), explain why the result of (b) was to be expected.

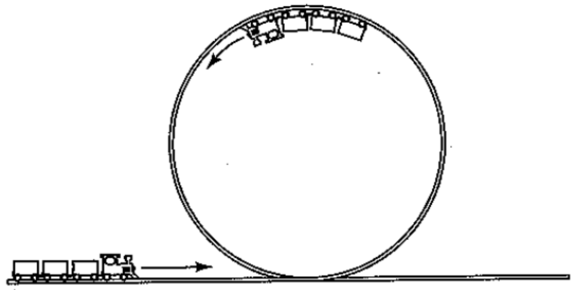
Problem 4 (20 points) Find the perturbative solution to the following equations, (i) up to the first order in λ, (ii) up to the second order in λ, and (iii) up to the third order in λ, assuming $|\lambda| \ll 1$.

(a) $x = 1 + \lambda\sqrt{2 - x}$

(b) $x - \frac{\pi}{4} = 2\lambda \cos x$

Useful formulae: $\sqrt{1+\delta} = 1 + \frac{1}{2}\delta - \frac{1}{8}\delta^2 + \frac{1}{16}\delta^3 + O(\delta^4)$, $\cos(a + b) = \cos a \cos b - \sin a \sin b$, $\cos\delta \approx 1 - \frac{\delta^2}{2} + O(\delta^4)$, $\sin\delta \approx \delta + O(\delta^3)$. where $|\delta| \ll 1$ (δ, not x).

Problem 5 [Provide your full analysis of the following problem as presented in the Phys/Math questionnaire.] A toy train travels around a loop-the-loop track. Is there a normal force exerted by the track on the train at the instant the train is at the top of the loop? If there is (in some cases), why? If there is not (in some cases), why not?



Problem 6 [Provide a full solution to the "monkey problem" in the Phys/Math questionnaire.] A monkey clings to a rope that passes over a frictionless pulley. The monkey's weight is balanced by the mass m of a block hanging at the other end of the rope. Both monkey and block are motionless. In order to get to the block, the monkey climbs a distance L (measured along the rope) up the rope. (a) Does the block move as a result of the monkey's climbing? (b) If so, in which direction and by how much?