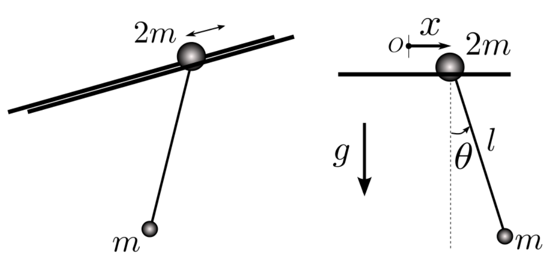
Two letter-size crib sheets are allowed. No calculator. No other materials.

$$\sin\delta = \delta + O(\delta^3),$$
$$\cos\delta = 1 - \frac{1}{2}\delta^2 + O(\delta^4).$$

Take time to read each problem with care. (And ask questions.) Be neat in writing. Answers without explanations may get you very few points only.
Believe that you are the one!! Good luck!

Problem 1 (100 points) Two (point) masses m and $2m$ are connected by a massless rod of length l. The two masses have negligible sizes and so they can be considered point masses to a good approximation, as we assume here. The mass $2m$ is placed on a rail, and it slides horizontally without friction. There is a constant downward gravitational field g. We take the coordinate system such that the y axis corresponds to the vertical direction while the x axis corresponds to the direction of the rail, i.e., the direction along which $2m$ slides freely. The motion of m is confined to the xy plane (cf., the side view diagram below).



Simple pendulum connected to a sliding mass (no friction)

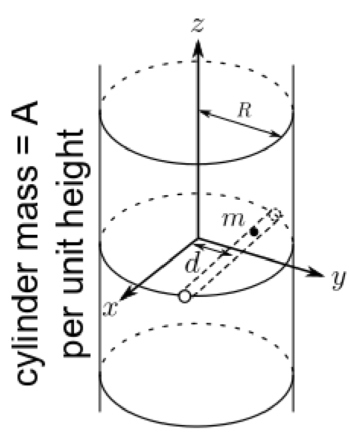
Side view

(a) Find the normal mode frequencies and the corresponding normal mode eigenvectors (or, equivalently, sketch each normal mode). Assume small angle displacement.

(b) Initially, the two masses and the rod are not in motion. A tiny ball of mud, of mass m, flies in the horizontal direction with speed v_0, and hits, and sticks to, the rod. Is it possible for the system (two masses + rod + mud) to show *only* a uniform translational motion after the mud sticks to the rod? If yes, where along the rod should the mud hit and what is the final speed of the system? If no, why not? You must explain your answer fully using force and torque due to impulse.

Problem 2 (100 points) A planet of mass m is orbiting around a heavy star of mass M. $M \gg m$ so that we can assume that the center of mass of the system is at the center of the star to a good approximation. The trajectory of the planet around the star is given by a circle. Suddenly, a mysterious internal explosion causes the planet to split into two pieces with mass $m_1 = \frac{m}{3}$ and $m_2 = \frac{2m}{3}$. It is found that the orbit of the first piece with mass m_1 is exactly the same as the initial circular orbit, except that the sense of rotation is reversed. Find the shape of the orbit of the second piece and answer whether m_2 will ever come back. The gravitational force between m_1 and m_2 is to be ignored.

Problem 3 (150 points) An infinitely long solid cylinder has mass A per unit height. Its density is uniform and its radius is given by R. The axis of the cylinder is defined as the z axis. Note that in this problem the cylinder is the only source of gravity.



(a) Find the gravitational field $\vec{g}$ due to this cylinder, inside and outside the cylinder. The use of the cylindrical coordinate system (in which $\rho = \sqrt{x^2 + y^2}$) is recommended.

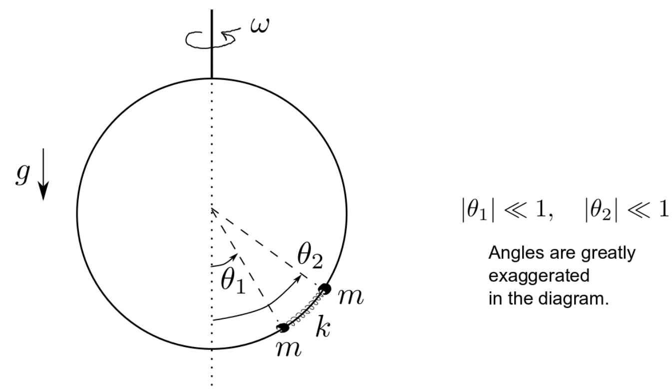
(b) Find the gravitational potential Φ corresponding to $\vec{g}$, inside and outside the cylinder. You must consider the continuity of the potential.

(c) A small hole is drilled so that a tunnel is created as shown in the diagram. This tunnel has negligible effect on the gravitational field generated by the cylinder due to the smallness of the diameter of the tunnel. The tunnel is parallel to the x axis, while it does *not* pass through the z axis. Instead, it crosses the y axis at distance d $(0 < d < R)$ from the origin. A point mass m is found with zero velocity at one end of the tunnel. What type of motion does this mass experience subsequently?

(d) Find the time for the mass to reach the other end of the tunnel.

(e) Does the mass m experience any other force (e.g., a constraint force) than the gravitational force during the motion inside the tunnel? If yes, find it. If no, explain why not.

Problem 4 (150 points) Along a circle shaped wire, two beads slide without friction. The wire has radius R and is mounted vertically. There is a constant downward gravitational field g. Each bead has mass m. The two beads are connected by a spring which wraps around the wire. The spring has zero mass and it causes no friction whatsoever. The equilibrium length of the spring is *zero*. The system is rotated, by an external force, around the vertical diameter of the circular wire at a *constant* angular velocity ω at all times. We shall consider only small angle motions, in which both θ_1 and θ_2 are small. Also, note that $\theta_2 > \theta_1$ by definition.



(a) Find the Lagrangian of the system. The generalized coordinates are suggested to be θ_1 and θ_2, the angular positions of the two beads on the circle.

(b) Show explicitly that this problem separates into two independent problems, i.e., $L_{tot} = L_M + L_{int}$, where L_{tot} is the total Lagrangian (which you found in the previous part), L_M is the Lagrangian for the average coordinate of the two beads, and L_{int} is the Lagrangian for the internal motion of the two-beads-on-spring system. You must find L_M and L_{int} correctly. Suggested generalized coordinates: $Q = (\theta_1 + \theta_2)/2$ and $q = \theta_2 - \theta_1$.

(c) Find the Hamiltonians corresponding respectively to L_{tot}, L_M, and L_{int}, and determine whether any of these Hamiltonians is conserved or not.

(d) Find the generalized momentum associated with L_M and discuss whether it is conserved. Repeat the same for L_{int}.

(e) Is the concept of effective potential applicable to L_M or L_{int}? If your answer is yes, find U_{eff}. If no, explain.