

Overarching rule --- No contact, no force
 (except gravity, magnetic force, ...)
 (static, van-der-Waals, ...)

Lagrangian mechanics

• $L = T - U$

• T in polar coord. spherical coord.

• $-\frac{\partial L}{\partial q} + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$ ----- (*)

• Constraint(s) $a_t + \sum_i a_i \dot{q}_i = 0$

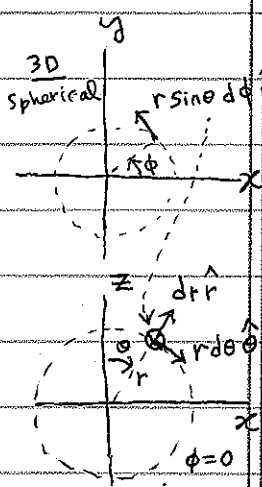
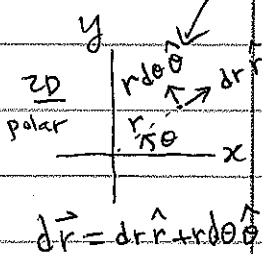
$-\frac{\partial L}{\partial q_i} + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = a_i \lambda$ or $\sum_s a_{s,i} \lambda_s$
 force of constraint if multiple constraints

• Conservation principles (no constraints)

$H = \sum_i p_i \dot{q}_i - L$ conserved if no explicit t dependence in $L = L(q_i, \dot{q}_i, t)$ $\rightarrow i$ collective index
 $L \neq L(\dot{q}_i) \Rightarrow p_i = \frac{\partial L}{\partial \dot{q}_i}$ is conserved. L (or H)
 no explicit dependence on q_i [from (*)] \rightarrow particular i

• Closed system --- $\vec{P}^{\text{total}}, \vec{L}^{\text{total}}, H = E$ conserved.

$M \vec{R}^{\text{total}} \parallel \vec{L}_M + \vec{L}_{\text{int,cm}} \parallel T_M + T_{\text{int,cm}}$



$d\vec{r} = dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi}$

• $dV = r^2 \sin\theta dr d\theta d\phi$

• $\vec{v} = \frac{d\vec{r}}{dt} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \sin\theta \dot{\phi} \hat{\phi}$

• $v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2$

"Jacobian" = $r^2 \sin\theta$

• $U_{\text{eff}}?$ H (or E) when conserved = $K + U_{\text{eff}}(q)$
 kinetic energy $\leftarrow U + \text{centrifugal term etc.}$
 (for one \dot{q})

Why a big deal? $\rightarrow K = A \dot{q}^2, U_{\text{eff}} = U_{\text{eff}}(q)$
 $(A = \frac{1}{2} m)$ \rightarrow can solve for $q(t)$ or rather $t(q)$

Central force problem

- $U_{\text{eff}}(r) = ?$ When does a circular motion occur?
 - Angular momentum conservation --- Kepler's 2nd
 - Kepler problem $U(r) = -\frac{k}{r}$
 - $\frac{d}{dt} = 1 + E \cos \theta$ $r^2 \propto a^3$ for elliptical orbits
 - 1st law $0 \leq E < 1$ 3rd law a : semi-major axis
 - Some general properties that can be "derived" or recalled from the circular motion?
 - centripetal force equation $\mu \frac{v^2}{R} = \frac{k}{R^2}$
 - E-conservation $\frac{1}{2} \mu v^2 - \frac{k}{R} = E$
 - $\rightarrow r^2 = \dots a^3$
 - $\rightarrow 2\langle T \rangle = -\langle U \rangle$
 - $\rightarrow \langle E \rangle = -\frac{k}{2a}$
- valid for elliptical orbits

Coupled oscillators

- $T = \frac{1}{2} \dot{\vec{q}}^T M \dot{\vec{q}}$ M, A : symmetric
- $U = \frac{1}{2} \vec{q}^T A \vec{q}$
- $(\overset{\leftarrow}{A} - \omega^2 \overset{\leftarrow}{M}) \vec{T} = 0 \rightarrow \vec{q} = \vec{T}_i \eta_i$
 - \uparrow eigenvalue ω_i
 - \uparrow eigenvector \vec{T}_i
 - for the i -th normal mode
 - $\eta_i = D_i \cos(\omega_i t + \phi_i)$
 - as many i 's as q_i 's
- \rightarrow diagonalize $\rightarrow \eta_i$'s (normal modes)
- $\vec{q} = \overset{\leftarrow}{T} \vec{\eta} = \begin{bmatrix} \vec{T}_1 & \dots & \vec{T}_n \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}$ $\vec{\eta} = \overset{\leftarrow}{T}^{-1} \vec{q}$
- $\overset{\leftarrow}{T}$ usually (almost) orthogonal.

other (more general) methods of finding the normal modes

① Assume $q_j = u_j e^{i\omega t}$ → solve for u_j 's
 or $e^{i\omega t}$ equivalent to calculating \vec{T}_i
 or \vec{T}

② By "intuition" or "experience" or "inspection of symmetry," one can write down solutions and then prove them.

--- Proving amounts to showing that each q_j oscillates with the same ω and obtaining ω . "Writing down solutions" amounts to writing down \vec{T} 's.

①, ② are more generally applicable to problems for which the " \vec{M}, \vec{A} method" is not applicable (e.g. when there is a damping).
 even ✓

For general problems, esp. rigid body problems

EOM set 1	$\begin{cases} \dot{\vec{P}} = \vec{F}_{ext} \\ \dot{\vec{L}}_{int,cm} = \vec{N}_{ext} \end{cases}$	These are valid even if the c.m. frame is generally not an inertial frame.
EOM set 2	$\begin{cases} \dot{\vec{P}}_M = \vec{F}_{ext} \\ \dot{\vec{L}}_{int,cm} = \vec{N}_{ext} \end{cases}$	
EOM set 3	$\dot{\vec{L}}_{total} = \vec{N}_{ext} \quad \text{in an inertial frame}$ <p style="text-align: center;"> \uparrow torque </p>	

NOTE: the inertia tensor will NOT be in the final exam.

Review - (X)

Rigid body

$$\vec{I} = \sum_i m_i \begin{bmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & z_i^2 + x_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{bmatrix}$$

$$\text{or } \int dm \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & z^2 + x^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$

Principal axes of inertia ... mutually orthogonal axes that diagonalizes

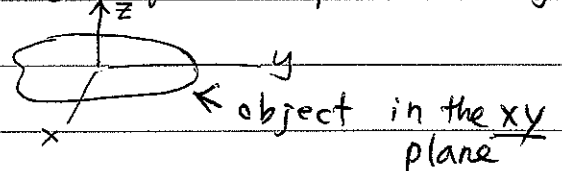


$$\vec{I} \rightarrow \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

We work with this as a rule.

Need to know how to identify them and then to calculate I_1, I_2, I_3 .

Helpful) ① $I_3 = I_1 + I_2$ for a planar object.



$$\vec{I} = \vec{I}_M + \vec{I}_{int,cm}$$

(parallel axis theorem) the "usual" \vec{I} in the cm frame

$$M \begin{bmatrix} y_{cm}^2 + z_{cm}^2 & -x_{cm} y_{cm} & -x_{cm} z_{cm} \\ -x_{cm} y_{cm} & z_{cm}^2 + x_{cm}^2 & -y_{cm} z_{cm} \\ -x_{cm} z_{cm} & -y_{cm} z_{cm} & x_{cm}^2 + y_{cm}^2 \end{bmatrix}$$

For rotations
 $\vec{L} = \vec{I} \vec{\omega}$
 $\vec{T} = \frac{1}{2} \vec{\omega} \cdot \vec{L}$
 $= \frac{1}{2} \vec{\omega} \cdot \vec{I} \vec{\omega}$

May Need to add \vec{T}_M or \vec{L}_M to get the total.

$$\left(\frac{d\vec{A}}{dt} \right)_{nr} = \left(\frac{d\vec{A}}{dt} \right)_r + \vec{\omega} \times \vec{A}$$



Derived from Eq. 10.1

Rolling w/o slipping : \vec{v} at contact point = 0 relative to the surface on which the thing rolls

Gravity (Newton's laws and Gauss laws)

$$\int d\vec{S} \cdot \vec{g} = -4\pi G M_{\text{inside}}$$

Review - ⑤

collisions

$$-\vec{\nabla}\Phi = \vec{g}, \quad \vec{F} = m\vec{g}$$

- Momentum conserved.
- Angular momentum conserved.

) for a closed system

for open system too, if external force is not large (\because collision time is short)

- Kinetic energy is generally not conserved. (inelastic collision)

Unless, it is said to be an elastic collision (or it is completely obvious), assume inelastic.

- Coefficient of restitution $\epsilon = \frac{|u_1 - v_2|}{|u_1 - u_2|}$ for head-on collision.
 u : initial
 v : final

- $T' (= T_{\text{int,cm}}) = \frac{1}{2} \mu |\dot{\vec{r}}_1 - \dot{\vec{r}}_2|^2$ is cool to remember.

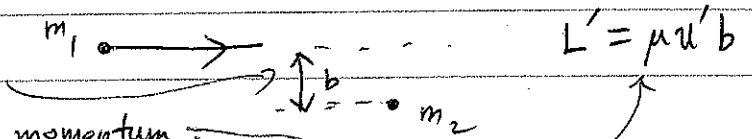
- On collision, a violent exchange of 3rd law forces impart impulse to a particle

$$\vec{I} = \vec{F}_{\text{ave}} \cdot \Delta t = \Delta \vec{p} \text{ for that particle}$$

The torque delivered to that particle

$$\vec{r} \times \vec{I} / \Delta t$$

$$\therefore \Delta \vec{L} = \vec{r} \times \vec{I}$$

- Impact parameter (b)  $L' = \mu u' b$
 related to angular momentum -
 reason for \odot non-head-on collisions.