- The following pages show the *types* of problems that are candidates for the final exam.

- However, the list is not exhaustive.

- To prepare for the final exam, you must (1) review all homework and quiz problems, and (2) go over all lecture notes.

Prob. 1 A particle in a central potential, $V(r)$, in three dimensions, can be described in terms of a one-dimensional problem with an effective potential, $U_{eff}(r)$,

$$U_{eff}(r) = V(r) + \frac{L^2}{2mr^2}$$

where L is the angular momentum.

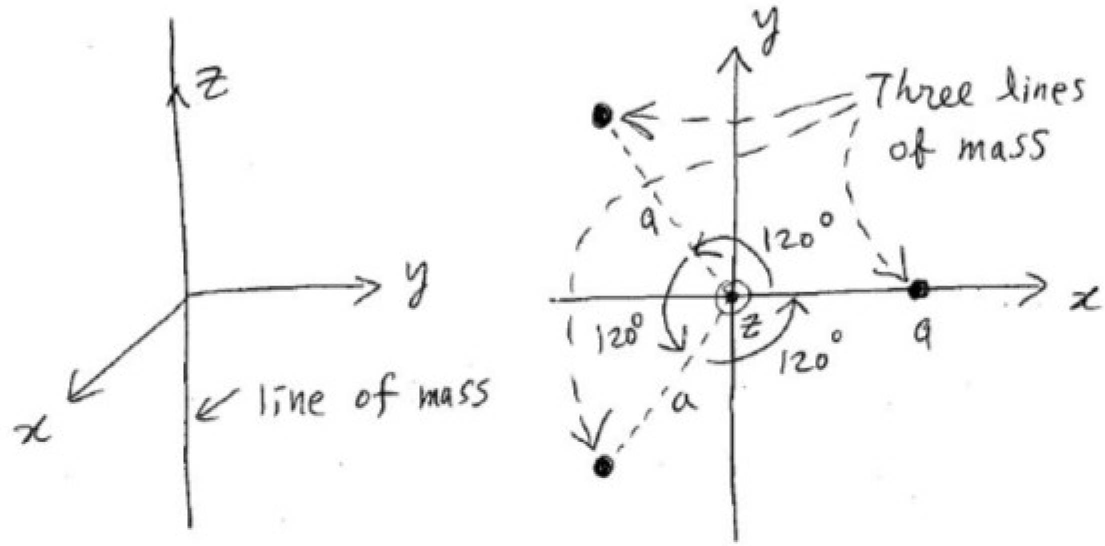
a. Derive this expression. (Hint: you may want to keep in mind that motion in a central potential is in a plane).

b. For a *circular* orbit, determine r for a potential $V(r) = Cr^x$, for a given value of L. (If $x > 0$, $c > 0$; if $x < 0$, $c < 0$).

c. What is the condition on x for the stability of the orbit?

Prob. 2 This problem consists of three parts. If you are not able to do part (a), then you may still do parts (b,c) assuming point masses instead of line masses.

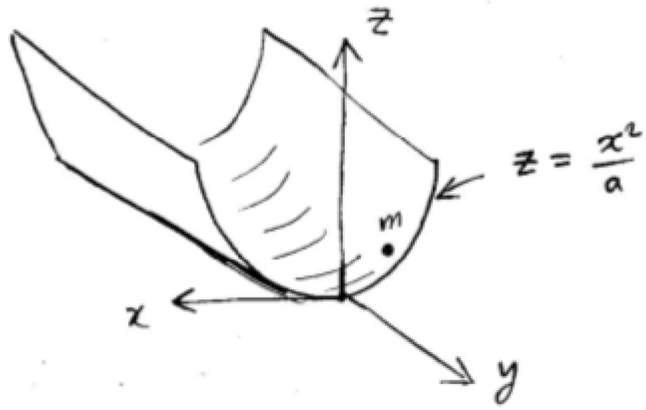
(a) There is a line of mass along the z axis, passing through the origin (the left figure below). The total mass M and the total length of the line L are infinite, but the mass per unit length is finite, $A = M/L$. Find the gravitational potential Φ, in terms of A and position coordinates. [Hint: Gauss law, $\int d\vec{S} \cdot \vec{g} = -4\pi GM$.]



(b) There are three lines of mass, each identical with the one in (a) but placed at different positions. The top view is depicted in the right figure above (no line of mass at the origin – the circle and the dot at the origin are just for the z axis.) Find the gravitational potential Φ due to them.

(c) Expand the potential of (b), to 2nd order in x and y, assuming $|x|, |y| \ll a$.

Prob. 3 A particle with mass m moves, without friction, on a "trough" whose shape is given by $z = x^2/a$. There is no friction. There is a constant gravitational field, $\vec{g} = -g\hat{z}$. The following diagram shows a bottom section of the trough. Note that the trough is infinite in length (in y) and height (in z).



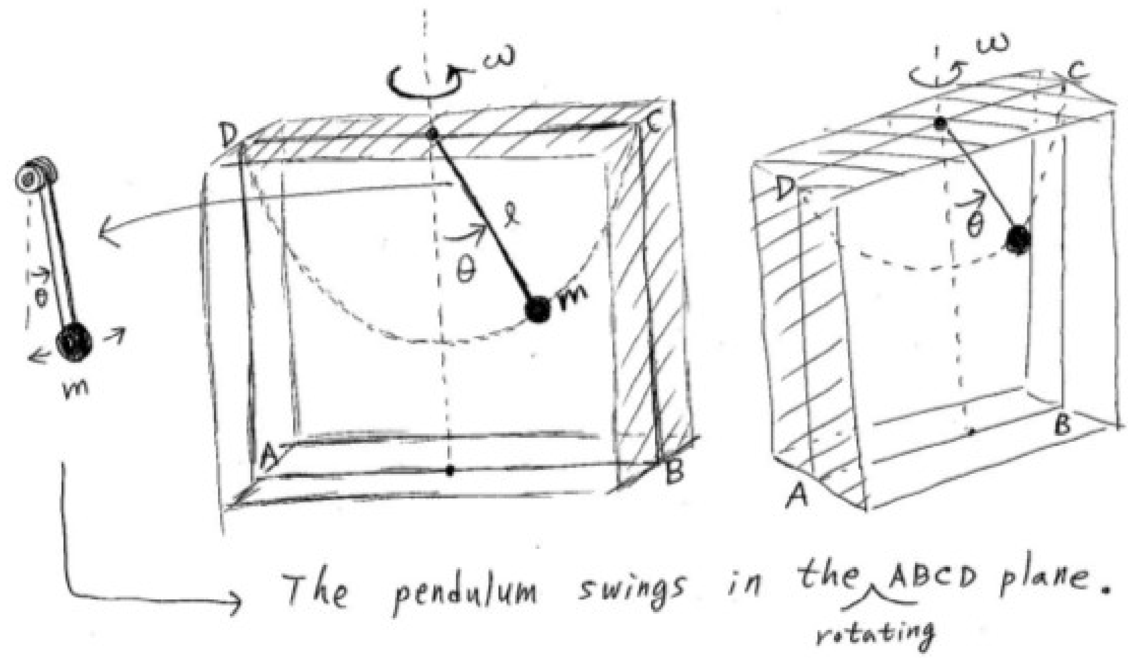
(a) Find the Lagrangian.

(b) Find the equations of motion for x and y.

(c) For small x, find the period of the x motion in terms of g, a. [Hint: $\dot{x}$ or $\ddot{x}$ is $O(x)$. Ignore terms that are higher order than linear in x in the equation of motion. For example, $x^2\ddot{x}$ would be such a term ($O(x^3)$) to ignore.]

(d) What are the conserved quantities among H, p_x, p_y, L_z? (H: Hamiltonian, p_x, p_y: linear momentum along x and y respectively, L_z: the z-component of the angular momentum.) Concisely explain why.

Prob. 4 A simple plane pendulum is attached to the center of the ceiling of a room. The pendulum is constrained to oscillate in the plane ABCD. The pendulum swings without any damping. The room is rotating around the vertical axis at a constant angular velocity ω, as does the plane ABCD in which the pendulum oscillates. Note that $|\theta| < \pi/2$.



(a) Find the Lagrangian.

(b) Find the canonical momentum p_θ and the Hamiltonian H.

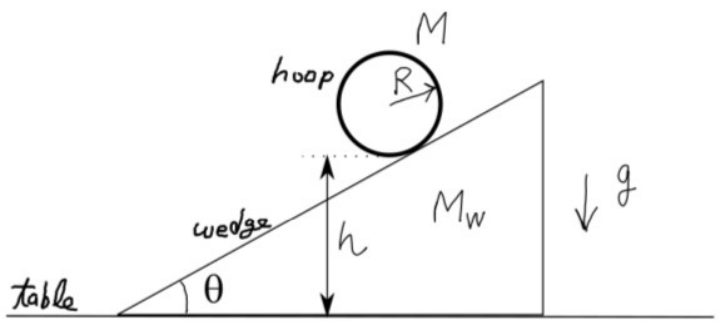
(c) Which of the quantities H, E, p_θ is/are conserved? For each quantity, explain briefly why it is conserved (or not). Here, $E = T + U$, where $T = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)$.

(d) In the reference frame of the room, what is the effective potential, $V(\theta)$, for this pendulum, given the above Lagrangian and Hamiltonian?

(e) Find all possible equilibrium points for $V(\theta)$. Show that some equilibrium points exist only when $\omega \geq \omega_c$. Find ω_c.

(f) Sketch $V(\theta)$, in the two regimes, $\omega < \omega_c$ and $\omega > \omega_c$. [Hint: If you are unsure how to proceed, then it may help to first examine the limits $\omega \to 0$ and $\omega \gg \omega_c$. Using the physical intuition, rather than the mathematical rigor, is allowed for producing the sketch.]

Prob. 5 A hollow cylinder ("hoop") with radius R and total mass M is placed on a wedge of mass $M_W = 3M$ and incline angle θ, as shown in the figure. The constant gravitational field g points downward. The wedge is *free to slide on the table without friction.*



Initially, the cylinder is at height h, measured from the table to the lowest point of the cylinder, and zero total kinetic energy. The wedge also has zero kinetic energy, initially. The cylinder is then released and starts to roll down. The wedge slides. We consider the motion between $t = 0$ and $t = t_t$ only, where $t = 0$ is the time of release, and t_t is the time at which the cylinder touches the table. Assume that the cylinder rolls without slipping.

(a) Find the acceleration of the cylinder relative to the wedge, and relative to the table.

(b) Find t_t. Find the velocity of the center of mass of the cylinder and the velocity of the wedge, both at $t = t_t$ and both relative to the table. This part may be done after doing part (c), if you like.

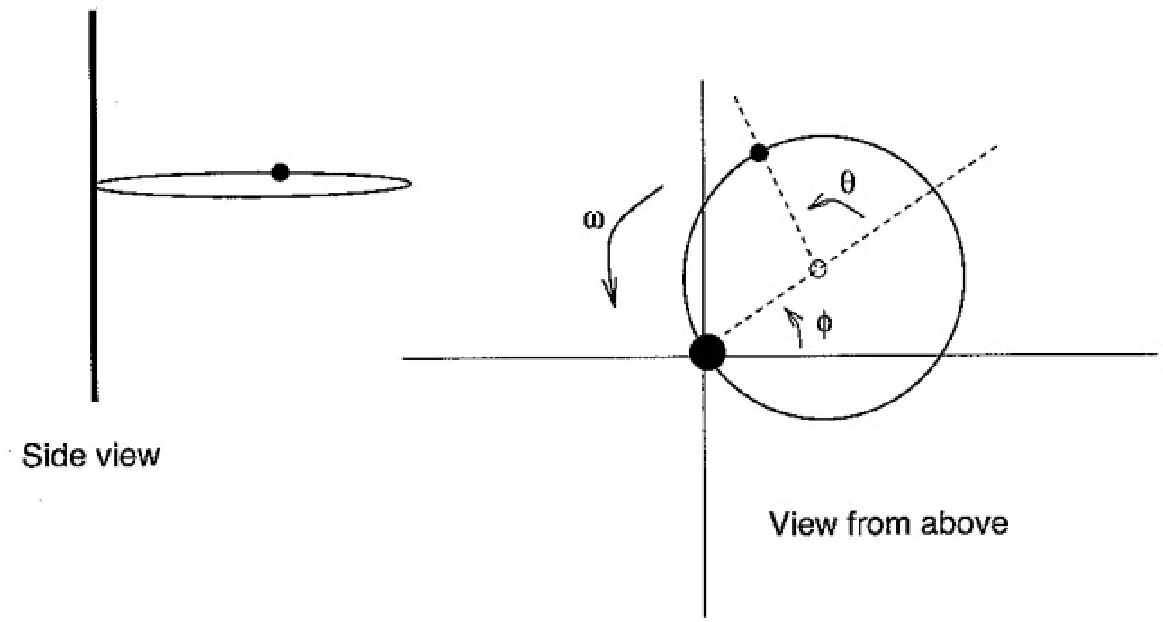
(c) Find, and discuss the nature of, all constants of motion of this problem.

(d) Find the force of constraint that acts along the surface of the wedge that the cylinder rolls on. Be specific as to which object exerts this force on which object. Find the total work done by this force of constraint from $t = 0$ to $t = t_t$.

(e) Consider a solid cylinder, with the same mass M, but with radius r, which may not be equal to R. This solid cylinder is given the same initial condition and travels down the same height h before touching the table. Would t_t(solid) for this case be greater than or less than t_t(hollow) of the above case? Explain your answer, as precisely as possible, but avoiding any unnecessary calculation. Do *not* assume that the formula for the rotational inertia for the solid cylinder is known. If you think it is necessary to know it, then you should derive it.

Prob. 6

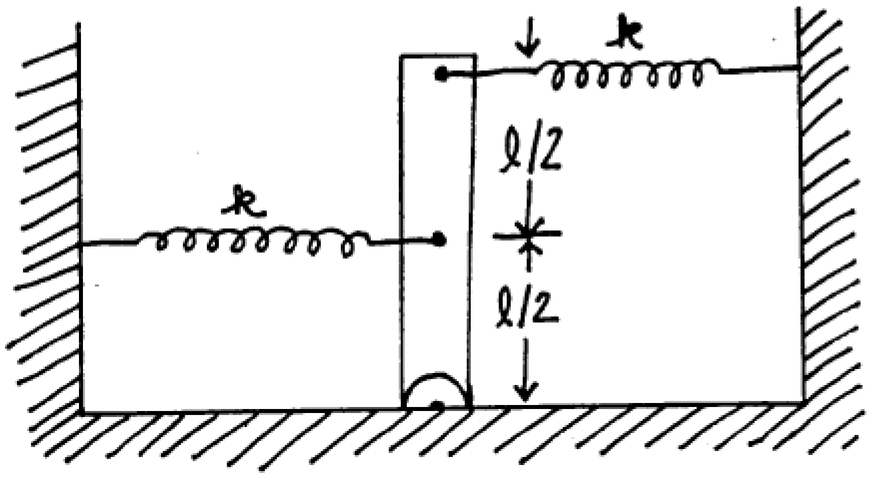
A bead of mass m slides frictionlessly on a wire loop of radius R. The loop is attached at one point to a vertical rotating rod ($\phi = \omega t$) so that the loop rotates in the horizontal plane about a point on its rim.



(A) Derive the equation of motion for the angle θ that the bead makes relative to the diameter pointing toward the rod (see figure).

(B) Find an equation for the radial force constraining the bead to on be the loop, as a function of θ, $\dot{\theta}$, and the constants m, R, and ω.

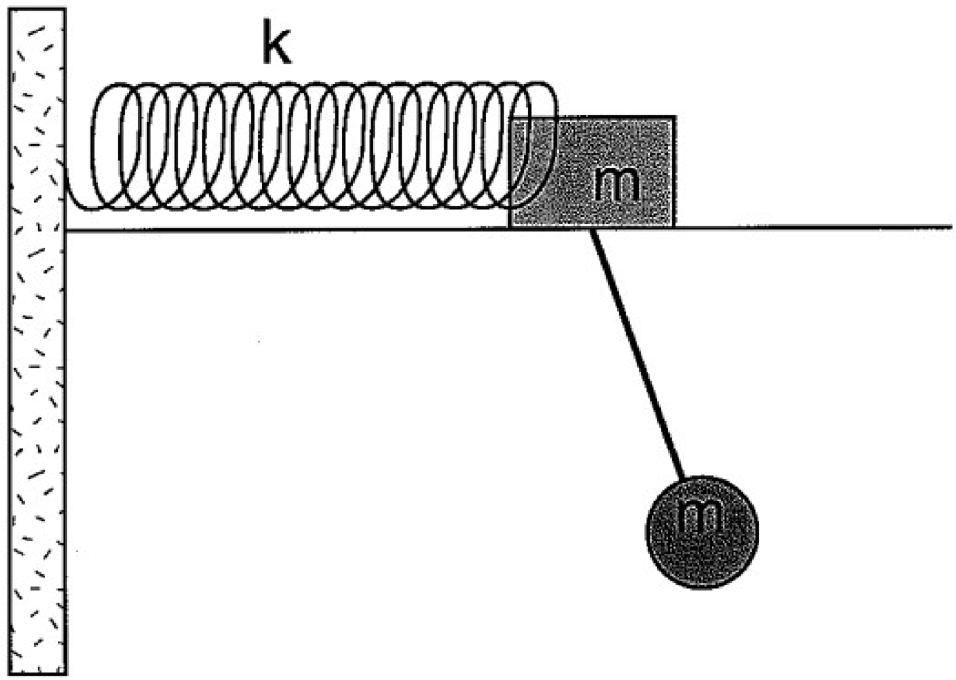
Prob. 7



A rod of length l and mass M pivots at one end. It is held by two springs one at the top and the other at the mid height. Both springs have spring constant k and pull perpendicular to the rod at vertical equilibrium. What is the angular frequency for small oscillations about the equilibrium position? Gravity can be neglected.

Prob. 8

A block of mass m is connected to a massless spring of spring constant k such that it can oscillate horizontally. From the block is suspended a simple plane pendulum of length ℓ and mass m. Assume that all of the motion takes place in a single plane with no friction.



(A) Choose appropriate generalized coordinates and express the kinetic energy T in terms of them.

(B) Express the potential energy U in terms of the generalized coordinates.

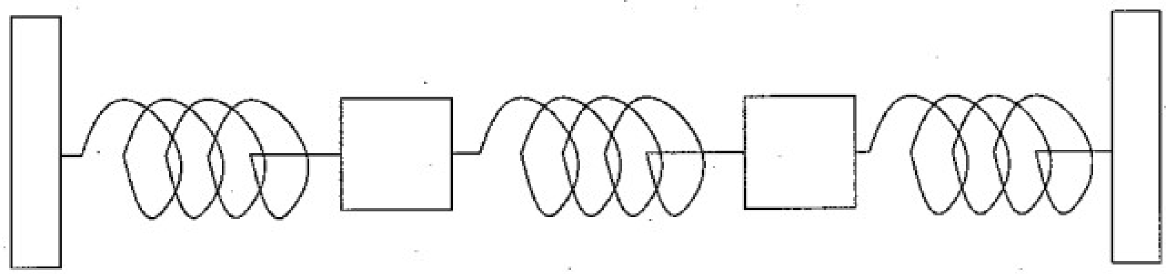
(C) Derive the equations of motion from Lagrange's equations.

(D) Expand the kinetic and potential energies about the equilibrium point to arrive at the constant tensors m and A such that $T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k$ and $U = \frac{1}{2}\sum_{j,k} A_{jk}q_jq_k$.

(E) Solve for the two eigenfrequencies of small oscillations. Do not find the corresponding eigenvectors.

Prob. 9

Consider the linear system of masses and springs shown below. The spring constants for the two outer springs are g, while the inner spring has spring constant h. The masses of the blocks are equal (m).



(A) Suppose that both masses start from their equilibrium positions. If the left-hand mass has an initial velocity v_0 toward the right-hand mass, calculate the position of the right-hand mass as a function of time. (At some point in your solution, describe clearly and succinctly the basic approach and procedure you are following.)

(B) Suppose that the left-hand mass is displaced a distance x_0 away from its equilibrium position toward the right-hand mass. Calculate the position of the right-hand mass as a function of time.

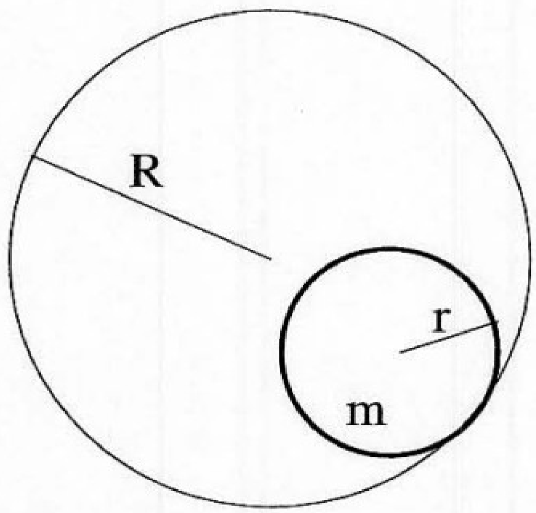
Prob. 10 A hollow cylinder of circular cross section, radius r and mass m rolls, under the force of gravity, without friction and without slipping inside a fixed cylinder of circular cross section and radius R.

a. Write the Lagrangian for this system.

b. Find the equation of motion.

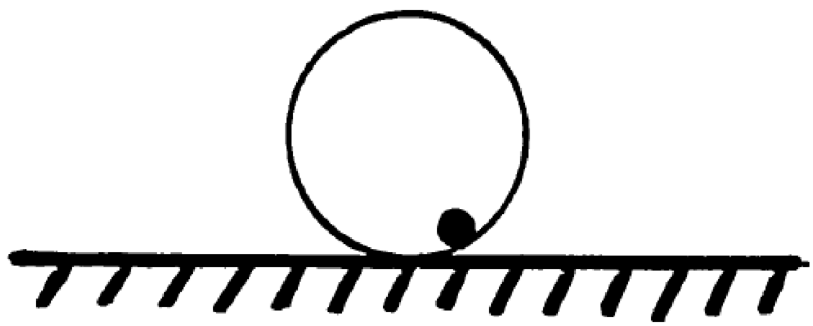
c. Find the frequency of small oscillations.

d. Let us say that we require for the small cylinder to reach the top without falling. Find the minimum speed of the center of mass of the small cylinder when it is at the bottom, to meet this requirement.



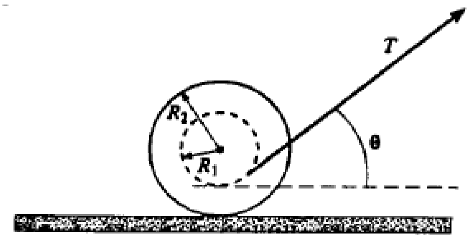
Prob. 11

A point mass m is attached to the rim of an otherwise uniform hoop of mass M and radius R. Calculate the frequency of small-amplitude rolling oscillations.



Prob. 12

A spool rests on a rough table as shown. A thread wound on the spool is pulled with force T at angle θ.



a) Show that there is an angle θ for which the spool remains at rest.

b) At this critical angle find the maximum T for equilibrium to be maintained. Assume a coefficient of friction μ.

Prob. 13

A billiard ball is a spherical object with uniform density. The rotational inertia is given by $2MR^2/5$.
You hit it with a cue. Assume that the cue is moved along the horizontal direction only, and so the applied force is horizontal. Where should you be hitting in order to minimize the energy loss of the ball as it rolls without slipping or slides while rolling? Assume that you can hit anyhwere, not just on the vertical line of symmetry. [Hints: The minimum energy loss occurs for rolling without slipping. The answer is not a point, but a line.]

Prob. 14

Find the normal modes of the following coupled oscillator system, assuming small oscillations.

Here the system consists of four identical beads of mass m that can slide without friction on two stiff and smooth perpendicular wires, which lie fixed in space. There is no gravity in this problem. Each neighboring pair of beads are connected by a spring of spring constant k. In addition, two diagonal springs with spring constant κ connect the two diagonal pairs of beads, as shown. [These diagonal springs are mounted in a way (e.g. one below the wire and the other above the wire) so that they do not touch each other. They also exert no friction on wires.] In equilibrium, the four beads form a square shape. For each normal mode, find the normal mode frequency, the $\vec{T}$ vector, and the diagram of the normal mode motion. As always, in particular for this one, answers without explanations may get you very few points, if any.